Charge Density in a Current-Carrying Wire

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(December 23, 2010)

1 Problem

Discuss the volume densities ρ_+ and $\rho_- < 0$ of positive and negative electric charges in a wire that carries a steady current, assuming that in the lab frame the positive charges are at rest and the current is due to negative charges (electrons) that all have speed v.

2 Solution

Discussions of the force on a charged particle outside a current-carrying wire often assume that the wire is electrically neutral. This problem explores how this assumption is not quite correct.

We give solutions both in the lab frame and in the rest frame of the conduction electrons, using both Maxwell's equations and special relativity.

We note that for current to flow in a resistive wire, there must be an axial electric field inside the wire, which requires a surface charge distribution that varies with position along the wire. See, for example, sec. 17 of [1], and [2, 3]. The surface charge distribution could include a uniform term of any magnitude. These surface charges are kept from leaving the surface by quantum effects often summarized by the term "work function." Likewise, the positive charges in the interior of the wire are held together in a lattice by quantum effects. We suppose that the positive charge density ρ_+ is uniform in the lab frame.

In this problem we assume that the conduction electrons can be described classically. Then, for steady axial motion, there must be zero radial force on these electrons.

We use a cylindrical coordinate system (r, ϕ, z) whose axis is the axis of the wire. We suppose that the flow of conducting electrons is purely axial, and azimuthally symmetric. The negative charge density, ρ_-, could depend on the radius r. The electric field $\mathbf{E}$ has no azimuthal component, while the magnetic field $\mathbf{B}$ has only an azimuthal component.

2.1 Lab Frame

For steady flow of current, the axial component of the electric field must be independent of z. Then, assuming that there is no azimuthal component to the electric field, Maxwell's first equation tells us that

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{1}{r}\frac{\partial}{\partial r}(rE_r) = 4\pi\rho(r) = 4\pi[\rho_+ + \rho_-(r)] \tag{1}$$

(in Gaussian units). The radial component of the electric field vanishes at $r = 0$, so we find that

$$E_r(r) = \frac{4\pi}{r}\int_0^r r'[\rho_+ + \rho_-(r')]\, dr' \tag{2}$$

(in Gaussian units).

The azimuthal magnetic field is due to the motion of the negative charges with velocity $v\,\hat{\mathbf{z}}$, and Ampère's law tells use that

$$B_\phi(r) = \frac{4\pi v}{cr} \int_0^r r' \rho_-(r')\, dr'. \tag{3}$$

The Lorenz force density on the negative charges is

$$\begin{aligned} \mathbf{f}_-(r) &= \rho_-(r) \left(\mathbf{E}(r) + \frac{\mathbf{v}}{c} \times \mathbf{B}(r) \right) \\ &= \rho_-(r) \left\{ \frac{4\pi\,\hat{\mathbf{r}}}{r} \int_0^r r' \left[\rho_+ + \rho_-(r') \left(1 - \frac{v^2}{c^2} \right) \right] dr' + E_z(r)\,\hat{\mathbf{z}} \right\}. \end{aligned} \tag{4}$$

For steady axial motion of the conduction electrons, the radial force on them must vanish at all r inside the wire, which implies that the negative charge density is uniform with value

$$\rho_- = -\frac{\rho_+}{1 - v^2/c^2} = -\gamma^2 \rho_+, \tag{5}$$

where

$$\gamma = \sqrt{\frac{1}{1 - v^2/c^2}}\,. \tag{6}$$

The positive charge density is less than the negative by one part in 10^{21} for $v = 1$ cm/s. This corresponds roughly to 10 more electrons than protons in each cubic millimeter of copper wire.

Using eq. (5) in eqs. (2)-(3), the radial electric field and azimuthal magnetic field inside the wire are

$$E_r(r) = -2\pi(\gamma^2 - 1)\rho_+ r, \qquad B_\phi(r) = -\frac{2pi\gamma^2 v \rho_+ r}{c}\,. \tag{7}$$

Because of the small difference between the positive and negative charge densities, the positive charges experience a small inward radial force density (in addition to the axial force due to collisions with the conduction electrons),

$$f_{+,r} = \rho_+ E_r(r) = -2\pi(\gamma^2 - 1)\rho_+^2 r = -\frac{2\pi\gamma^2 v^2 \rho_+^2 r}{c^2}\,. \tag{8}$$

2.2 Rest Frame of the Conduction Electrons

We denote quantities in the rest frame of the conduction electrons with a *. Thus, the velocity of the charged particles in the * frame is $\mathbf{v}_+^\star = -v\,\hat{\mathbf{z}}$.

If the radial electric field $E_r^\star$ were nonzero the conduction electrons would experience a radial force. Hence, we expect the radial electric field, and the bulk charge density $\rho^\star$, to vanish in the * frame.

The Lorentz contraction of a moving "stick" results in an observer of a "charge stick" reporting a charge density larger by a factor γ than that in the rest frame of the "stick." Thus, the density ρ_- of negative charge in the lab frame is larger than that in the $\star$ frame,

$$\rho_- = \gamma \rho_-^\star, \tag{9}$$

while the density $\rho_+^\star$ of positive charge in the $\star$ frame is larger than that in the lab frame,

$$\rho_+^\star = \gamma \rho_+. \tag{10}$$

The total charge density in the $\star$ frame is

$$\rho^\star = \rho_+^\star + \rho_-^\star = \gamma \rho_+ - \frac{\gamma^2 \rho_+}{\gamma} = 0. \tag{11}$$

Thus, the bulk charge density of a current-carrying wire vanishes in the rest frame of the conduction charges, rather than in the lab frame [4].

The positive charges experience a magnetic Lorentz force density in the $\star$ frame,

$$f_{+,r}^\star = -\frac{\rho_+^\star v B_\phi^\star}{c} = -\frac{2\pi {\rho_+^\star}^2 v^2 r}{c^2} = -\frac{2\pi \gamma^2 v^2 \rho_+^2 r}{c^2} = f_{+,r}. \tag{12}$$

This transverse Lorentz force is the same in the lab and the $\star$ frames, as expected for the transverse spatial component of a 4-vector.[1]

References

[1] A. Sommerfeld, *Electrodynamics* (Academic Press, New York, 1952).

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[4] P.C. Peters, *In what frame is a current-carrying conductor neutral?*, Am. J. Phys. **53**, 1165 (1985),
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[1]The Lorentz force $\mathbf{f} = \rho\mathbf{E} + \mathbf{J}/c \times \mathbf{B}$ on a charge-current density 4-vector $j_\mu = (\rho, \mathbf{J}/c)$ is the spatial component of the 4-vector $f_{\mathrm{Lorentz},\mu} = (\mathbf{J}\cdot\mathbf{E}/c, \mathbf{f})$. Thus, the Lorentz force density is "more relativistic" than the so-called Minkowski force based on the Newtonian relation $\mathbf{F} = m\mathbf{a}$, for which the 4-vector relation is $f_{\mathrm{Minkowski},\mu} = (\gamma\mathbf{F}\cdot\boldsymbol{\beta}, \gamma\mathbf{F})$ for the force on a particle of velocity $\mathbf{v} = \boldsymbol{\beta}c$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$, in that the Minkowski force has the awkward factor of γ in its spatial components. Of course, the (Minkokwski) force $\mathbf{F} = \mathbf{f}$ dVol on a charged volume element, dVol, is not a component of a 4-vector; however, $\gamma\mathbf{F} = \mathbf{f}\gamma$ dVol is.