

# The Electromagnetic Fields Outside a Wire That Carries a Linearly Rising Current

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## 1 Problem

A neutral wire along the  $z$ -axis carries current  $I$  that varies with time  $t$  according to

$$I(t) = \begin{cases} 0 & t \leq 0, \\ \alpha t & t > 0, \end{cases} \quad \alpha \text{ is a constant.} \quad (1)$$

Deduce the time-dependence of the electric and magnetic fields,  $\mathbf{E}$  and  $\mathbf{B}$ , observed at a point  $(r, \theta = 0, z = 0)$  in a cylindrical coordinate system about the wire. Use your expressions to discuss the fields in the two limiting cases that  $ct \gg r$  and  $ct = r + \epsilon$ , where  $c$  is the speed of light and  $\epsilon \ll r$ .

## 2 Solution

We follow the familiar method of first calculating the retarded potentials and then taking derivatives to find the fields. The retarded scalar and vector potentials  $V$  and  $\mathbf{A}$  are given by

$$V(\mathbf{x}, t) = \int \frac{\rho(\mathbf{x}', t - R/c) d^3\mathbf{x}'}{R}, \quad \text{and} \quad \mathbf{A}(\mathbf{x}, t) = \frac{1}{c} \int \frac{\mathbf{J}(\mathbf{x}', t - R/c) d^3\mathbf{x}'}{R}, \quad (2)$$

in Gaussian units, where  $\rho$  and  $\mathbf{J}$  are the charge and current densities, respectively, and  $R = |\mathbf{x} - \mathbf{x}'|$ .

In the present case, we assume that the wire remains neutral when the current flows.<sup>1</sup> Then, the scalar potential vanishes. For the vector potential, we see that only the component  $A_z$  will be nonzero. Also,  $\mathbf{J} d^3\mathbf{x}'$  can be rewritten as  $I dz$  for current in a wire along the  $z$ -axis. For an observer at  $(r, 0, 0)$  and a current element at  $(0, 0, z)$ , we have  $R = \sqrt{r^2 + z^2}$ . Further, the condition that  $I$  is nonzero only for time  $t > 0$  implies that it contributes to the fields only for  $z$  such that  $(ct)^2 > R^2 = r^2 + z^2$ . That is, we need to evaluate the integral only for

$$|z| < z_0 \equiv \sqrt{(ct)^2 - r^2}. \quad (3)$$

Altogether,

$$A_z(r, 0, 0, t) = \frac{\alpha}{c} \int_{-z_0}^{z_0} \left( \frac{t}{\sqrt{r^2 + z^2}} - \frac{1}{c} \right) dz = \frac{\alpha}{c} \left( t \ln \frac{ct + z_0}{ct - z_0} - \frac{2z_0}{c} \right) = \frac{2\alpha}{c} \left( t \ln \frac{z_0 + ct}{r} - \frac{z_0}{c} \right). \quad (4)$$

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<sup>1</sup>We ignore the small departure from neutrality that scales as  $v^2/c^2$  where  $v \approx 1$  cm/sec is the velocity of the conduction electrons.

[The two forms in eq. (4) arise depending on whether or not one notices that the integrand is even in  $z$ .]

The magnetic field is obtained via  $\mathbf{B} = \nabla \times \mathbf{A}$ . Since only  $A_z$  is nonzero, the only nonzero component of  $\mathbf{B}$  is (noting that  $\partial z_0 / \partial r = -r / z_0$ )

$$B_\phi = -\frac{\partial A_z}{\partial r} = \frac{2\alpha z_0}{cr}. \quad (5)$$

The only nonzero component of the electric field is

$$E_z = -\frac{1}{c} \frac{\partial A_z}{\partial t} = -\frac{2\alpha}{c^2} \ln \frac{z_0 + ct}{r}. \quad (6)$$

For long times,  $ct \gg r$ ,  $\Rightarrow z_0 \approx ct$ , and the fields become

$$B_\phi \approx \frac{2\alpha ct}{cr} = \frac{2I(t)}{cr} = B_0(t), \quad E_z \approx -\frac{2\alpha}{c^2} \ln \frac{2ct}{r} = -B_0 \frac{r}{ct} \ln \frac{2ct}{r} \ll B_0, \quad (7)$$

where  $B_0(t) = 2I(t)/cr$  is the instantaneous magnetic field corresponding to current  $I(t)$ . That is, we recover the magnetostatic limit at large times.

For short times,  $ct = r + \epsilon$  with  $\epsilon \ll r$ , after the fields first become nonzero we have

$$z_0 = \sqrt{2r\epsilon + \epsilon^2} \approx \sqrt{2r\epsilon}, \quad (8)$$

so

$$B_\phi \approx \frac{2\alpha}{c^2} \sqrt{\frac{2\epsilon}{r}}, \quad \text{and} \quad E_z \approx -\frac{2\alpha}{c^2} \ln \frac{r + \epsilon + \sqrt{2r\epsilon}}{r} \approx -\frac{2\alpha}{c^2} \sqrt{\frac{2\epsilon}{r}} = -B_\phi. \quad (9)$$

In this regime, the fields have the character of radiation, with  $\mathbf{E}$  and  $\mathbf{B}$  of equal magnitude, mutually orthogonal, and both orthogonal to the line of sight to the closest point on the wire. (Because of the cylindrical geometry the radiation fields do not have  $1/r$  dependence – which holds instead for cylindrical static fields.)

In sum, the fields build up from zero only after time  $ct = r$ . The initial fields propagate outwards at the speed of light and have the character of cylindrical waves. But at a fixed  $r$ , the electric field dies out with time, and the magnetic field approaches the instantaneous magnetostatic field due to the current in the wire.<sup>2</sup>

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<sup>2</sup>Of possible amusement is a direct calculation of the vector potential for the case of a constant current  $I_0$ . First, from Ampere's law we know that  $B_\phi = 2I_0/cr = -\partial A_z/\partial r$ , so we have that

$$A_z = -\frac{2I_0}{c} \ln r + \text{const.} \quad (10)$$

If we use the integral form for the vector potential we have

$$A_z(r, 0, 0) = \frac{1}{c} \int_{-\infty}^{\infty} \frac{I_0 dz}{\sqrt{r^2 + z^2}} = \frac{2I_0}{c} \int_0^{\infty} \frac{dz}{\sqrt{r^2 + z^2}} = -\frac{2I_0}{c} \ln r + \lim_{z \rightarrow \infty} \ln(z + \sqrt{z^2 + r^2}). \quad (11)$$

Only by ignoring the large constant, which does not depend on  $r$  for a long wire, do we recover the “elementary” result.