Testing Inflation: A Bootstrap Approach

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We note that the essential idea of inflation, that the universe underwent a brief period of accelerated expansion followed by a long period of decelerated expansion, can be encapsulated in a "closure condition" which relates the amount of accelerated expansion during inflation to the amount of decelerated expansion afterward. We present a protocol for systematically testing the validity of this condition observationally.

Is it possible to show convincingly that inflation [1, 2] is responsible for the large-scale homogeneity, isotropy and flatness of the universe, and the primordial spectrum of metric fluctuations that seeded galaxy formation and sourced the temperature and polarization variations in the cosmic microwave background (CMB)? Some would claim no, because there is too much freedom in constructing inflationary models. Even if one shows that the observations are consistent with the predictions of a particular inflationary model, this is unconvincing because, for virtually any given combination of observations, one can design many inflationary models that reproduce them. If a theory allows everything, it has no predictive power.

In this paper, though, we describe how to combine observations into a sequence of "bootstrap tests" that, if any one of them is passed, will be the most direct confirmation possible that the universe underwent a brief period of acceleration ($d^2a/dt^2 > 0$) followed by a long period of deceleration ($d^2a/dt^2 < 0$). [Here $a(t)$ is the Robertson-Walker scale factor and $H(t) \equiv d(\ln a)/dt$ is the Hubble rate [2].] Let us briefly summarize the basic idea, with details postponed until the next section. First, note that $a(t)H(t)$ grows during acceleration, and shrinks during deceleration. According to the inflationary scenario [2], any observable fourier mode of the cosmological density field with comoving wavenumber k_* reached a moment during inflation known as "horizon exit," at which the ratio $a(t)H(t)/k_*$ was unity; then, during the period remaining before the end of inflation, this ratio grew by $N_{\rm bef}(k_*)$ e-folds; and finally, after inflation, the ratio shrunk by $N_{\rm aft}$ e-folds, reaching the value of $a_0 H_0/k_* < 1$ measured today. Thus, in this picture, the mode k_* must satisfy the "closure condition"

$$\ln(a_0 H_0/k_*) = N_{\rm bef}(k_*) - N_{\rm aft}. \qquad (1)$$

How can we test this equation? If we know the temporal evolution of H, from the moment that k_* leaves the horizon until the end of inflation, we have enough information to compute $N_{\rm bef}(k_*)$ and $N_{\rm aft}$ and check that Eq. (1) is correct. Unfortunately, cosmological observations will never give us this much information; instead, assuming inflation is correct, they will provide the first few terms in a Taylor approximation to H

around the moment that the "WMAP wavenumber" [3] $k_* = 0.002$/Mpc left the horizon. Imagine we only have enough observations to determine this Taylor approximation up to jth order: this is our best guess for H given the available data. Under the assumption that this guess remains valid all the way to the end of inflation, we can check whether Eq. (1) is true. If it is, then we not only have evidence for our guess, but for the idea of inflation on which it is based: we have pulled ourselves up by our bootstraps! Note that $a_0 H_0/k_*$, as measured today, is exponentially sensitive to both $N_{\rm bef}(k_*)$, which depends on the expansion history *during* inflation, and $N_{\rm aft}$, which depends on the expansion history *after* inflation. Thus, if observations can be used to determine all three quantities and if they are shown to satisfy the closure condition, even an ardent skeptic would be hard-pressed to discount it as coincidence; it would be strong evidence for inflation, and a tall challenge for any competing theory. And if there are any doubters, the bootstrap method offers in some cases a series of follow-up checks that can turn a convincing verification into an overwhelming one.

As discussed in the conclusion, the closure test also has the advantage that it is experimentally easier to apply compared to other proposed inflationary tests, such as the "consistency relations" [4, 5]. As for failing the closure test, this does not mean inflation is ruled out, because it is always possible to construct inflationary models that fit the data; as precision improves, observers can continue to distinguish viable and non-viable models. However, in this sad circumstance, cosmological observations will probably never yield the kind of convincing confirmation of inflation discussed here.

KEY OBSERVABLES AND EQUATIONS

Before describing the bootstrap tests, let us introduce the key observables, equations and parameters we will need. From $\Delta_{\mathcal{R}}^2(k)$ and $\Delta_h^2(k)$, the scalar and tensor power spectra, one defines the tensor/scalar ratio $r \equiv \Delta_h^2/\Delta_{\mathcal{R}}^2$, the scalar index $n_s - 1 \equiv d(\ln \Delta_{\mathcal{R}}^2)/d(\ln k)$, its "running" $\alpha_s \equiv dn_s/d(\ln k)$, the "running of the running" $\beta_s \equiv d\alpha_s/d(\ln k)$, and so on, all measured at the

wavenumber k_* at which they are most precisely determined. We follow the standard WMAP definitions for all of these observables [3]; and for concreteness fix $k_* = 0.002$/Mpc following WMAP [3].

We focus first on the case where the Hubble expansion rate H during inflation is governed by a single order parameter that acts just like a single inflaton scalar field, φ [6, 7] [15] (Generalizations will be discussed in the concluding section.) For a clear derivation and presentation of all of the equations in this section, see Liddle *et al* [6]. Subscripts "$*$" and "end" will be used to indicate that the corresponding quantity is to be evaluated at the moment when k_* exits the horizon, or at the end of inflation, respectively. Without loss of generality, we can choose $\varphi_* = 0$, and Taylor expand:

$$H(\varphi) = H_* + H'_* \varphi + \frac{1}{2} H''_* \varphi^2 + \frac{1}{6} H'''_* \varphi^3 + \dots \quad (2)$$

If, again without loss of generality, we take $d\varphi/dt > 0$ (or, equivalently, $H'_* < 0$), and choose "reduced Planck units" with $\hbar = c = 8\pi G = 1$, then the first few coefficients are

$$H_* = \frac{\pi(\Delta_\mathcal{R}^2)^{1/2}}{2} (2r)^{1/2}, \quad (3a)$$

$$H'_* = \frac{\pi(\Delta_\mathcal{R}^2)^{1/2}}{8} (-r), \quad (3b)$$

$$H''_* = \frac{\pi(\Delta_\mathcal{R}^2)^{1/2}}{32} (2r)^{1/2}[r + 4(n_s - 1)], \quad (3c)$$

$$H'''_* = \frac{\pi(\Delta_\mathcal{R}^2)^{1/2}}{128} [64\alpha_s - 3r^2 - 20r(n_s - 1)]. \quad (3d)$$

The end of inflation ($H = H_{end}$ and $\varphi = \varphi_{end}$) occurs when $\ddot{a} = 0$ or, equivalently,

$$H_{end} = -\sqrt{2}\, H'_{end}. \quad (4)$$

Finally, in the closure condition (1), we have

$$N_{\text{bef}}(k_*) = \ln(H_{end}/H_*) - \frac{1}{2} \int_{\varphi_*}^{\varphi_{end}} d\varphi [H(\varphi)/H'(\varphi)] \quad (5)$$

and

$$N_{\text{aft}} = \ln[\Omega_{\text{rad}}^{1/4}(H_{end}/H_0)^{1/2}] + \Delta N, \quad (6)$$

where Ω_{rad} is the current ratio of the radiation density to the critical density and

$$\Delta N \equiv (1/12) \left[(1 - 3w_{re})/(1 + w_{re}) \right] \ln(\rho_{re}/\rho_{end}) \quad (7)$$

represents the uncertain physics of the epoch between the end of inflation and the start of radiation domination: w_{re} is the effective equation-of-state during this epoch, and ρ_{re} is the energy density at the start of radiation domination. We first consider the case where $\Delta N \approx 0$, which corresponds to "efficient" reheating ($w_{re} \approx 1/3$ or $\rho_{re} \approx \rho_{end}$). However, the uncertainty in ΔN, does not seriously interfere with the bootstrap test. To illustrate the point, we let ΔN be a free parameter, subject only to the weak assumptions that $0 \leq w_{re} \leq 1/3$ and $\rho_{\text{bbn}} \leq \rho_{re} \leq \rho_{end}$, where $\rho_{\text{bbn}} \approx (1\,\text{MeV})^4$ is roughly the energy density during big bang nucleosynthesis (BBN); then $\Delta N \leq 0$.

THE BOOTSTRAP TESTS

The bootstrap test uses precise observations at $k = k_*$ to obtain progessively better estimates of $H(\varphi)$ and H_{end}, which are, then, applied to determine if the closure condition is satisfied. If we regard $\Delta_\mathcal{R}^2 = (2.45 \pm 0.1) \times 10^{-9}$ [3] as an already-measured quantity, then the Taylor expansion (2, 3) of $H(\varphi)$ organizes the remaining observables into an ordered list $\{r, n_s, \alpha_s, \beta_s, \dots\}$ in the sense that, if we imagine that we only know the first j items in this list, then we can only determine the Taylor expansion up to jth order [16]. This is the best guess for $H(\varphi)$ based on the available data; using it, H_{end} is computed from Eq. (4); $N_{\text{bef}}(k_*)$ and N_{aft} are determined from Eqs. (5, 6); and finally the closure condition (1) is checked.

The jth bootstrap test is satisfied if the first j observables satisfy the closure condition. In practice, only the first three observables $\{r, n_s, \alpha_s\}$ can be detected or constrained tightly enough to be relevant for confirming inflation. Therefore, the first three bootstrap tests are the relevant ones, for all practical purposes: let us describe them and explain how they may be confirmed and cross-checked with forthcoming observations.

First bootstrap test. To start, imagine we are only given the first observable, r, so our best guess for $H(\varphi)$ is $H_* + H'_* \varphi$. We introduce this into Eq. (4) to obtain $\varphi_{end} = 2^{1/2}[(16/r)^{1/2} - 1]$, and apply these expressions for $H(\varphi)$ and φ_{end} to Eqs. (5, 6) to obtain $N_{\text{bef}}(k_*)$ and N_{aft}. Then, the closure condition, Eq. (1), is satisfied if

$$r(\Delta N) = 8/[A + \Delta N + 1/2] \quad (8)$$

where $A \equiv \ln(a_0 H_0/k_*) + \frac{1}{4}\ln(8\Omega_{\text{rad}}\pi^2\Delta_\mathcal{R}^2/H_0^2) \approx 61$. This corresponds to $r = 0.13$ if $\Delta N = 0$, and $0.13 < r < 0.17$ if the uncertainty in ΔN is included.

If observations pass this first bootstrap test, it will be a remarkable success for the inflationary paradigm, and one that can be checked: since true success should not be spoiled by the next observable, n_s, we expect $n_s = 1 - r/4$ (so that $H''_* \approx 0$). If this follow-up test is *also* successful, then it should not be spoiled by the next observable, α_s: thus we expect $\alpha_s = [3r^2 + 20r(n_s - 1)]/64$ (so that $H'''_* \approx 0$). If observations pass the first bootstrap test (1), plus the two follow-up tests, it will be overwhelming evidence for a period of inflationary expansion.

Second bootstrap test. If the first bootstrap test fails, proceed to the second. Given the measured values of $\{r, n_s\}$, the best guess for $H(\varphi)$ is now $H_* + H'_* \varphi + \frac{1}{2} H''_* \varphi^2$. We introduce this expression into (4) and find the smallest positive root: $\varphi_{end}(r, n_s)$. Next these formulae for $H(\varphi)$ and φ_{end} can be used in Eqs. (5, 6) to find $N_{\text{bef}}(k_*)$ and N_{aft}. Finally, substituting all of this into the closure condition (1), we obtain a relation between r, n_s corresponding to the solid black curve in Fig. 1; allowing for the uncertainty in ΔN thickens the curve to the shaded region in the figure. Passing the second
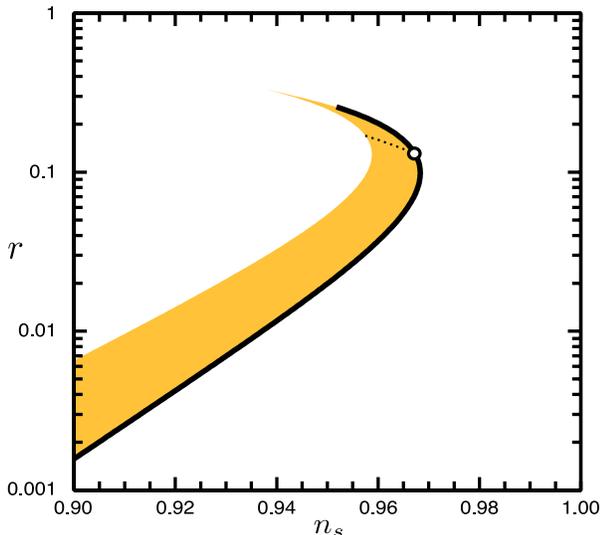
FIG. 1: First and second bootstrap tests: The first test is satisfied for r and n_s corresponding to the open white circle for efficient reheating; the circle becomes a short arc (dotted) if one includes the uncertainty in the "reheating epoch" (parameterized by ΔN). The second test is passed if r and n_s lie anywhere along the solid black curve or, allowing for the uncertainty in ΔN, the black curve plus shaded region.

bootstrap test would be nearly as remarkable as passing the first; and, here again, the success can be verified by measuring the next observable, α_s, and finding that $\alpha_s = [3r^2 + 20r(n_s - 1)]/64$ (so that $H_*''' \approx 0$).

Third bootstrap test. If the first two bootstrap tests fail, proceed to next order. Use the first three observables $\{r, n_s, \alpha_s\}$ to determine the coefficients in $H(\varphi) = H_* + H_*'\varphi + \frac{1}{2}H_*''\varphi^2 + \frac{1}{6}H_*'''\varphi^3$. Introduce this into Eq. (4) to find the smallest positive root $\varphi_{end}(r, n_s, \alpha_s)$ and use the expressions for $H(\varphi)$ and φ_{end} in Eqs. (5, 6) to compute $N_{bef}(k_*)$ and N_{aft}. Finally, substitute these expressions into Eq. (1) to check the closure condition. If $\Delta N = 0$, the closure condition will be satisfied for a 2-dimensional surface in the 3-dimensional space parametrized by $\{r, n_s, \alpha_s\}$: several surface contours are shown in Fig. 2. Allowing for the uncertainty in ΔN thickens this surface into a "thin slab" (or, equivalently, each curve in Fig. 2 extends *downward* to form a strip). In the limit of *extremely* small r, this slab has a simple analytic description: $\varphi_{end}(r, n_s, \alpha_s)$ is given by $\varphi_{end} = (-2/\alpha_s)^{1/2} r^{1/4}$, and $r = r(n_s, \alpha_s, \Delta N)$ satisfies

$$r = 8 \exp\left[\frac{8\pi - 16\arctan[(1-n_s)/y]}{y} - 4(A + \Delta N)\right] \quad (9)$$

where $y \equiv [-4\alpha_s - (n_s - 1)^2]^{1/2}$.

We can make α_s as negative as possible (for fixed n_s) by first letting r be as small as possible [for illustration, let us take the relatively weak assumption $\rho_f > (1 \text{ TeV})^4$ and hence $r > 8 \times 10^{-55}$], and then letting ΔN be as

negative as possible ($w_{re} = 0$ and $\rho_{re} = \rho_{bbn}$). In this way, we find that, if $\{r, n_s, \alpha_s\}$ pass the third bootstrap test, then α_s has a lower bound [8] $\alpha_s > \alpha_s^{min}(n_s)$, where $\alpha_s^{min}(n_s)$ varies smoothly from $\alpha_s^{min} = -0.0094$ (for $n_s = 0.9$) to $\alpha_s^{min} = -0.0161$ (for $n_s = 1$).

We have seen that, if the first bootstrap test is passed, then n_s and α_s provide two cross-checks; and, similarly, if the second bootstrap is passed, then α_s provides a single cross-check. But if the third bootstrap relation correctly predicts that the $\{r, n_s, \alpha_s\}$ lie in the slab described above, there will be no analogous cross checks available, since we will have used up our observables (see below for caveats). Nevertheless, passing the third test is an impressive verification of the inflationary principle.

DISCUSSION

Observations will measure r and n_s to a precision of roughly ± 0.01 in this decade, and perhaps ± 0.001 eventually. The first bootstrap test would give the most impressive proof of inflation, since it makes the largest number of verifiable follow-up predictions; the ranges $0.13 < r < 0.17$ and $n_s = 1 - r/4$ still agree well with current observations [3], but will either be confirmed or ruled out within the next few years. If the first bootstrap test fails, the second may be passed for a wider range of r; but since it relates r to n_s, the allowed range of r may be restricted by constraining n_s. For example, if $n_s > 0.94$, as suggested by WMAP5 [3], then the second bootstrap test only requires searching for $r > 0.01$, and thus may also be completed over the coming decade. And then, if CMB polarization experiments determine that $r < 0.01$, all is not lost: the third bootstrap test may still be passed, but only if α_s has a substantially *negative* value (see Fig. 2) – *e.g.* negative enough to be detected by a proposed high-redshift galaxy survey designed to measure $|\alpha_s| \sim 0.001$ [9]. But if α_s is *too* negative, all three bootstrap tests fail: *e.g.* if $\alpha_s < -0.016$, then the tests fail for all $n_s < 1$, according to the discussion above.

Passing the bootstrap tests would be consistent with many of the most appealing and commonly arising inflationary models, with the fewest degrees of freedom, fewest parameters, and smoothest evolution [11]. For example, in Fig. 1, $V(\varphi) = m^2\varphi^2$ corresponds to the open circle (and hence passes the first bootstrap test); $V(\varphi) = \lambda\varphi^4$ corresponds to the upper endpoint of the solid curve; the symmetry-breaking (Higgs) potential $V(\varphi) = \Lambda^4[1 - (\varphi/\mu)^2]^2$ corresponds to the part of the solid curve *below* the open circle; and the pseudo-Nambu-Goldstone-boson (axion) potential $V(\varphi) = \Lambda^4[1 \pm \cos(\varphi/\mu)]$ lies within the shaded region. On the other hand, hybrid-like inflation models, including some of the widely discussed proposals motivated by string theory, would not pass the tests [10].

The bootstrap tests have the advantage that they can be performed with forthcoming data. Compare them
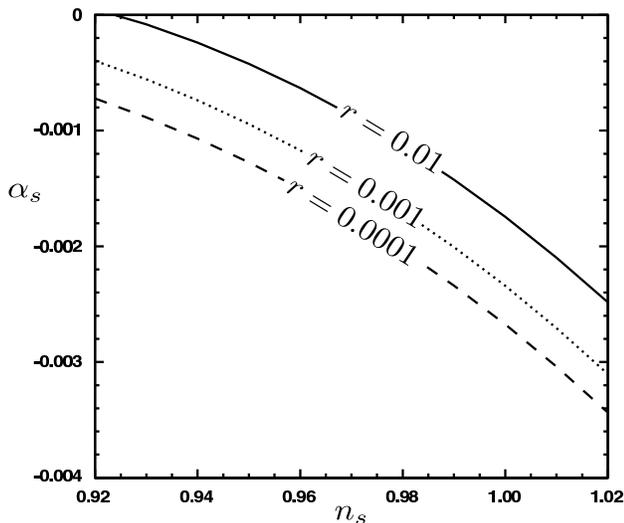
FIG. 2: The third bootstrap relation constrains (r, n_s, α_s) to lie on a curve, as shown, assuming efficient reheating ($\Delta N = 0$); including this uncertainty extends each curve downward (towards more easily observable values of α_s) into a strip of finite thickness. Note that these curves include values of (r, n_s) that fail the first two bootstrap tests.

with the well-known consistency relations for single field inflation: (i) a hierarchy of relations between the scalar and tensor power spectra [4]; and (ii) a hierarchy of relations between the primordial scalar N-point functions [5]. Confirming any of these relations requires measuring either (i) a non-zero value for the tensor spectral index n_t, or (ii) a non-gaussian primordial N-point function, both of which will be extremely difficult given the single-field inflationary predictions. Failure (detection of large deviations from the consistency relations) is observationally much easier than confirmation given the limitations of technology and foregrounds. By contrast, with the bootstrap tests, inflation can be precisely tested and confirmed using accessible technology and plausible foregrounds. (The consistency relations might eventually yield additional confirming tests.)

What if the bootstrap tests fail? Of course, there will still be some inflationary models and parameters that agree with the observations, and some that do not. But is there another way of confirming that inflation itself took place? Perhaps there is a generalized framework for inflation that gives rise to a generalized set of bootstrap tests which might still be passed? Interestingly, the two most common generalizations (allowing multiple order parameters [12, 13], or replacing the canonical kinetic term $X = \frac{1}{2}(\partial\varphi)^2$ by a general function of X [14]) give rise to frameworks that are *not* testable in our bootstrap sense: the observables $\{\Delta_\mathcal{R}^2, r, n_s, \alpha_s, \ldots\}$ do not intrinsically point to a best guess for both $N_{\rm bef}(k_*)$ and $N_{\rm aft}$. An exception is the subclass of single-field k-inflation models

[14] which *only* depend on X. In this case, a bootstrap test of (1) can be performed if one can *also* measure the tensor tilt n_t (although, as mentioned above, this is likely to be very difficult). An interesting corollary of our analyses is that, in all cases, a direct confirmation of inflation relies on being able to detect cosmic gravitational waves and measure accurately at least r.

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[15] For our purposes, using $H(\varphi)$ has an advantage over the other common alternative $V(\varphi)$. Namely, Eqs. (4,5,6), which play an important role in our analysis, are exact in terms of $H(\varphi)$, whereas the corresponding equations, when expressed in terms of $V(\varphi)$, invoke the slow roll approximation near the end of inflation, where it is expected to break down.
[16] Note that Taylor ordering $\{r, n_s, \alpha_s, \beta_s, \ldots\}$ is not the order in which these quantities are detected in practice: n_s has already been measured, but non-zero values for the remaining quantities (including r and α_s) have not yet been detected [3]. Still, the Taylor ordering is what is relevant for conducting bootstrap tests: for example, until one detects the first observable r, none of the bootstrap relations may be confirmed.