Inflationary predictions for scalar and tensor fluctuations reconsidered

Latham A. Boyle[1], Paul J. Steinhardt[1], and Neil Turok[2]

[1]*Department of Physics, Princeton University, Princeton, New Jersey 08544*
[2]*Department of Applied Mathematics and Theoretical Physics,*
Centre for Mathematical Sciences, University of Cambridge,
Wilberforce Road, Cambridge CB3 0WA, United Kingdom
(Dated: July 2005)

We reconsider the predictions of inflation for the spectral index of scalar (energy density) fluctuations (n_s) and the tensor/scalar ratio (r) using a discrete, model-independent measure of the degree of fine-tuning required to obtain a given combination of (n_s, r). We find that, except for cases with numerous unnecessary degrees of fine-tuning, n_s is less than 0.98, measurably different from exact Harrison-Zel'dovich. Furthermore, if $n_s \gtrsim 0.95$, in accord with current measurements, the tensor/scalar ratio satisfies $r \gtrsim 10^{-2}$, a range that should be detectable in proposed cosmic microwave background polarization experiments and direct gravitational wave searches.

Inflation predicts nearly scale-invariant spectra of primordial scalar (energy density) and tensor (gravitational wave) perturbations. What has been less clear is the precise prediction for the scalar spectral index n_s and the tensor/scalar ratio r. In particular, is n_s likely to be distinguishable from pure Harrison-Zel'dovich ($n_s = 1$)? And is r likely to be large enough for the tensor perturbations to be detected ($r \gtrsim 10^{-2}$)? One approach for addressing these questions is anecdotal experience based on explicitly constructing inflaton potentials $V(\phi)$ with different combinations of (n_s, r). A more recent approach is to use the inflationary flow equations to compute n_s and r for random choices of the "Hubble" slow-roll parameters [1], and plot the results as a dot-plot in the (n_s, r) plane. One problem with these methods is that the sampling does not incorporate a weight based on physical plausibility, so it does not provide a well-motivated measure of the relative likelihood across the (n_s, r)-plane. It is as if all inflaton potentials are created equal. Another problem in many of these studies is that only some of the minimal requirements for a successful inflaton potential are considered. It is simply assumed that the rest can be satisfied without reducing the attractiveness of the model. Yet, this assumption is often invalid in practice.

In this Letter, we attempt to rectify this situation by considering the complete set of inflationary conditions and introducing a discrete counting scheme for assessing the degrees of fine-tuning required to obtain a given combination of n_s and r. (An alternative approach is based on Bayesian model selection [2].) We find (see Fig. 1) that models with blue or slightly red tilts ($n_s > 0.98$) require significantly more degrees of fine-tuning than models with $n_s < 0.98$. This concurs with the intuitive impression obtained by trying to construct potentials by hand. More importantly, the procedure reveals valuable additional information: (1) a significant gap exists between the inflationary prediction for n_s and pure Harrison-Zel'dovich ($n_s = 1$), a difference that near-future measurements should be able to resolve; and (2) if $n_s \gtrsim 0.95$, as current measurements suggest, r exceeds 10^{-2}, so tensor fluctuations should be observable in proposed cosmic microwave background (CMB) polarization and gravitational wave interferometer experiments. (Interestingly, independent approaches based on physical field-theoretic arguments have reached similar conclusions [3].)

As we have emphasized, it is important to consider all the conditions necessary for inflation when assessing the degrees of fine-tuning, namely:

1. as ϕ evolves over some range $\Delta\phi$, the universe undergoes at least $N > 60$ e folds of inflation in order to become homogeneous, isotropic, spatially flat, and monopole-free;

2. after the field evolves past this range, inflation must halt and the universe must reheat without spoiling the large-scale homogeneity and isotropy;

3. the energy density (scalar) perturbations, which we assume are generated by the quantum fluctuations of the inflaton field, must have amplitude $\sim 10^{-5}$ on scales that left the horizon ≈ 60 e folds before the end of inflation, to agree with observations [4];

4. after inflation, the field must evolve smoothly (*i.e.*, without generating unacceptable inhomogeneities) to an analytic minimum with $V \approx 0$;

5. if the minimum is metastable, then it must be long-lived and V must be bounded below.

The analyticity condition is to avoid physically questionable terms of the form $|\phi|$ or ϕ^{α} where α is non-integer. Many analyses consider only the first three conditions, but we find that the fourth condition, which is equally essential, imposes a non-linear constraint on V that can significantly affect the degree of fine-tuning required to obtain a given (n_s, r). (We have stated the conditions above as if the inflaton potential is a function of a single field ϕ; the generalization to multiple fields is straightforward.)

To quantify the degree of fine-tuning, we count the number of unnecessary features introduced during the last 60 e folds of inflation to achieve a given (n_s, r). To

pose the conditions in a physically motivated and model-independent way, we use the standard slow-roll parameters:

$$\epsilon \equiv (3/2)(1+w) \approx (1/2)d\ln V/dN \qquad (1)$$
$$\eta \equiv (1/2)d\ln (V')^2/dN, \qquad (2)$$

where N is the number of e folds remaining before inflation ends and a prime indicates $d/d\phi$ for an inflaton field ϕ canonically normalized in Einstein frame. The parameters ϵ and η have a physical interpretation: they represent respectively the fractional rate of change of the Hubble parameter ($\propto V^{\frac{1}{2}}$) and the force on the scalar field ($\propto V'$) per inflationary e fold. In all inflationary models, ϵ and η must increase from small values ($\lesssim 1/60$) when $N \approx 60$ to values of order unity at the end of inflation ($N = 0$).

For minimally tuned models, the simplest being a monomial potential $V = \alpha\phi^n$ with integer n and a single adjustable coefficient α, both $\epsilon(N)$ and $\eta(N)$, as well as all of their derivatives (*e.g.*, $d^m\eta/dN^m$) are monotonic and have no zeroes during the last 60 e folds. The range of (n_s, r) associated with these models lies in the shaded region marked "0" in Fig. 1, which has $n_s < 0.98$ and $r > 10^{-2}$. To move further outside this range requires that more zeroes of η and its derivatives occur in the last 60 e folds. The zeroes are independent in the sense that they can be added one at a time by successively adjusting parameters, as shown in Fig. 1.

Our key point is this: As exemplified by the minimally tuned models, no zeroes whatsoever are required during the last 60 e folds to satisfy the five inflationary conditions. Hence, each zero added to the last 60 e folds can be properly construed as representing an extra degree of fine-tuning beyond what is necessary – an extra acceleration, jerk or higher order-shift in the equation of state (for ϵ) or the force (for η) artificially introduced at nearly the exact moment when the modes currently observed in the cosmic microwave background are exiting the horizon during inflation.

More specifically, the number of zeroes is a conservative (lower bound) measure of how many derivatives of $\epsilon(N)$ and $\eta(N)$ must be finely adjusted to achieve a given (n_s, r). This can be seen by constructing the Taylor expansion about $N_0 \approx 60$ and comparing the higher-order terms to the lower order ones at, say, $N = 10$. As the number of zeroes increases, more higher-order terms contribute non-negligibly before inflation ends, revealing the delicate toggling of the equation of state and the force V' during the last 60 e folds. Proceeding deeper into the gray region in Fig. 1 (many tunings), the terms in the Taylor series grow until the series is no longer absolutely convergent.

Therefore, we propose to quantify the fine-tuning by introducing the integers $Z_{\epsilon,\eta}$ which measure the number of zeroes that ϵ and η and their derivatives undergo within the last 60 e folds of inflation. Fig. 1 is based on zeroes of η; a similar result occurs for ϵ. We find
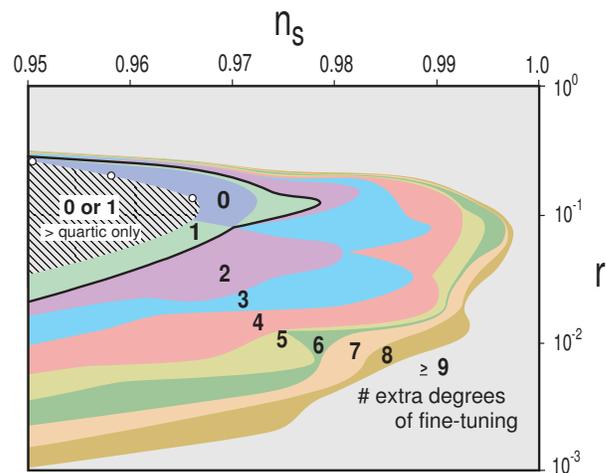

FIG. 1: Predictions for tensor/scalar ratio r versus spectral tilt n_s for minimal tuning ($Z_\eta = 0$) and for different degrees of extra fine-tuning ($Z_\eta \geq 1$). The small white circles correspond to monomial potentials (from right to left: quadratic, cubic, quartic). The thick curve enclosing all models with zero or one extra degree of fine-tuning has $n_s < 0.98$ and $r > 10^{-2}$; hatched portion has $Z_\eta = 0$ or 1 but is only accessible for polynomials of degree greater than four ($Z_{order} \geq 1$). Nine or more extra degrees of fine-tuning are required to obtain n_s close to 1 or small r (gray).

that these metrics are robust methods for dividing models into those that are simple (few zeroes) and those that are highly tuned (many zeroes). (N.B. The point of our metric is not to rank a model with $Z_\eta = 20$ over a model with $Z_\eta = 1000$; the significance of a difference in Z_η when Z_η is large for both models is unclear. Rather, Z_η is designed to show that both models are vastly more finely adjusted than models with $Z_\eta = 0$ or 1.)

Fig. 1 summarizes our analysis for quartic polynomial potentials $V(\phi)$ that satisfy the five inflationary conditions. The simplest models, the monomial potentials, are represented by a sequence of discrete white circles. Next, we consider more general polynomials combining terms of different order. The cases in which the coefficients all have the same sign lie on the boundary of the shaded region marked "0", along the curve connecting the white circles. All of these models are minimally tuned ($Z_\eta = 0$) and have $n_s < 0.97$ and $r > 10\%$.

A special case occurs among models with only one degree of fine-tuning $Z_\eta = 1$: namely, models tuned so that the 60 e-fold mark lies very near a maximum of the potential. Simple examples include the Mexican hat potential, $V(\phi) = V_0 - \frac{1}{2}m^2\phi^2 + \lambda\phi^4$ and the axion potential, $V(\phi) = V_0(1 + \cos(\phi/f))$. If ϕ at the 60 e-fold mark lies close to the maximum, then η has a zero since the force must have a maximum in these potentials. Although this kind of zero is unnecessary for inflation, it can occur naturally if the action is invariant under certain symmetries, as illustrated by the two examples above. Hence, we in-

clude this region within our thick black curve in Fig. 1. As the parameters of V are further adjusted so that ϕ lies very close to the maximum at the 60 e-fold mark, the allowed range in the (n_s, r) plane expands to fill out the shaded region marked "1."

Everywhere else in the (n_s, r) plane is reached by adding sequentially more zeroes of η and its derivatives within the last 60 e folds. Increasing the zeroes introduces one or more special features in V (extrema, inflection points, ...), progressively flattens the potential in the vicinity of the feature, and finely tunes ϕ at the 60 e-fold mark to lie closer and closer to it. Unlike the first $(Z_\eta = 1)$ case of tuning discussed above, there is no symmetry principle that dictates any of these additional tunings. Yet, as Fig. 1 shows, many such tunings are necessary to reach low values of r or high values of n_s.

Although Fig. 1 is based on quartic (renormalizable) polynomial potentials, a similar plot can be constructed for polynomials of arbitrary order. For polynomials of any order, there is always a wide range of parameters for which $Z_\eta = 0$ or 1, $n_s \lesssim 0.98$ and $r \gtrsim 10^{-2}$. With higher-order polynomials, it is possible to insert more bumps and jerks into the final 60 e folds, even though this is not required for inflation. Introducing them for the purpose of enabling anomalous values of n_s and r should be included in assessing the degrees of fine-tuning. Just as the zeroes are independent and can be added one by one, the space of polynomial functions can be extended order by order. Hence, we suggest amending the degrees of tuning to be $Z_\eta + Z_{order}$, where Z_{order} is the difference between the actual polynomial order and four. One regime that now becomes accessible with $Z_\eta = 0$ or 1 is the hatched region in Fig. 1; $n_s \gtrsim 0.98$ and/or $r \lesssim 10^{-2}$ (for $n_s \gtrsim 0.95$) still require many degrees of fine-tuning.

Models with more than one field can be treated in a similar way provided the path(s) describing the last 60 e folds of inflation and the passage to the potential minimum can be described by the classical equations of motion for the fields. We shall call these "deterministic." The above analysis may simply be applied to each path individually. As before, each path with many zeroes is related to paths with $Z_\eta = 0$ or 1 by a continuous fine-tuning of parameters. In some special models (like the hybrid model in [5]), the path is non-deterministic. Instead, the classical evolution reaches a critical point in the potential where quantum diffusion is needed to reach the end of inflation, as in the case $V = V_0 + \frac{1}{2}m_\phi^2\phi^2 - \frac{1}{2}m_\psi^2\psi^2 + \frac{1}{2}\gamma\phi^2\psi^2 + \dots$ (which has a critical point at $\gamma\phi_c^2 = m_\psi^2$ and $\psi = 0$). There is effectively a discontinuous jump in ϵ and η at the critical point, and there is not a unique procedure to relate these cases to models with $Z_\eta = 0$ or 1. As a result, counting zeroes may not be an appropriate way to judge them. We note that these cases include examples with $n_s \gtrsim 0.98$ and $r \lesssim 10^{-2}$; however, compared to the $Z_\eta = 0$ models in Fig. 1, one must add at least one extra field, an exponentially large mass hierarchy between m_ϕ and m_ψ, and,

for some (n_s, r), another hierarchy between the three dimensionless quartic couplings and/or one or more higher-order couplings [6].

Our conclusion that gravitational waves should be detectable runs contrary to some claims in the literature. A common but flawed argument has been that the amplitude of the tensor power spectrum is highly uncertain because it is proportional to the fourth power of the inflationary energy scale M_*, whose value is poorly determined. In actuality, although gravitational waves will allow us to determine M_*, their detectability only depends on the tensor/scalar ratio, r, which does not depend on M_* at all [7]. ($r \approx 16\epsilon$, so it only depends on the equation of state during inflation.)

A second argument by Lyth [8] and others makes the claim that r must be small in any theory which includes quantum gravity effects. They point out that the slow-roll equations imply the relation $\Delta\phi = (m_{Pl}/8\sqrt{\pi}) \int r^{1/2}dN$, where $m_{Pl} = 1.2 \times 10^{19}$ GeV is the Planck mass. From this relation, if $r \gtrsim .05$, then $\Delta\phi/m_{Pl}$ should exceed unity over the final 60 e folds of inflation. Lyth argues that once gravitational effects are included: (a) the effective potential V can only be reliably calculated over a domain $\Delta\phi < m_{Pl}$; and (b) inflation is likely to occur only over a range $\Delta\phi < m_{Pl}$. These two claims are disputed in [3], and several inflationary models [9] provide explicit counter-examples. These models have $\Delta\phi/m_{Pl} > 1$ and gravitational corrections are under control. They give values of n_s and r consistent with Fig. 1 and with current data.

Our analysis shows that forcing $\Delta\phi/m_{Pl}$ to be less than unity and maintaining a small number of zeroes requires $n_s \lesssim 0.95$, past the left boundary of Fig. 1 and outside the range favored by current data. To obtain $r < 10^{-3}$ and a small number of zeroes requires much smaller n_s. The only alternatives for obtaining small r are to introduce many zeroes or to turn to non-deterministic models. The latter, as we have noted, typically require extra fields and tunings compared to deterministic models with $Z_\eta = 0$ or 1.

The primordial spectrum of gravitational waves in Fig. 1 is related to the observable spectrum today (τ_0) by a "transfer function" $T_t(k, \tau_0)$ [10]:

$$\Omega_{gw}(k, \tau_0) \equiv \frac{1}{\rho_{cr}}\frac{d\rho_{gw}}{d\ln k} = \frac{1}{12}\frac{k^2}{a_0^2 H_0^2}T_t(k, \tau_0)\Delta_t^2(k), \quad (3)$$

where $\Delta_t^2(k)$ is the primordial tensor power spectrum, ρ_{gw} is the gravitational-wave energy density, ρ_{cr} is the critical density, and $\Omega_{gw}(k, \tau_0)$ is the ratio of the gravitational wave energy density in a log-interval about k to the critical density. We have used an analytic expression for the transfer function derived in [10] which improves on the accuracy of previous calculations [11] (but see also [12, 13]). The transfer function includes the redshift-suppression after horizon re-entry; the imprint of horizon re-entry itself; the possibility of dark energy with equation-of-state $w(z)$; the damping due to free-streaming relativistic particles (e.g., neutrinos) in the
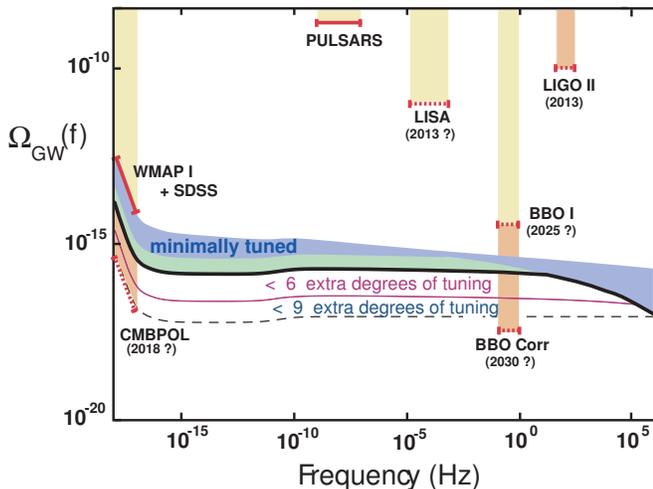
FIG. 2: Inflationary predictions of $\Omega_{gw}(f)$ vs. f with present (solid bars) and future (dashed bars) observational limits. The solid blue and green bands represent the predicted range for models with minimal tuning, or one extra degree of tuning, respectively, that satisfy the current observational bounds. The thick curve represents the lower-bound for Ω_{gw} from among the region enclosed by the black curve in Fig. 1. The purple curve is the lower bound for models with $Z_\eta < 6$. The dashed curve ($r = 10^{-3}$) is the lowest prediction among all models shown in Fig. 1.

early universe [14]; and several early-universe effects that were not considered in previous treatments.

Fig. 2 shows the inflationary predictions for Ω_{gw} as a function of frequency f compared to present and future observations, assuming current limits, from [15], on non-inflationary parameters. (For a related figure with the observations shown in more detail, see Fig. 2 in [12].)

The thick solid curve in Fig. 2 represents the lower bound among all models with minimal tuning ($Z_\eta = 0$) or one extra degree of fine-tuning ($Z_\eta = 1$). The thick dotted curve is the lower bound predicted for the entire range of models in Fig. 1; for quartic potentials, at least nine extra degrees of fine-tuning are required to go below it. Also notice the kinks near $f \sim 10^{-11}$ Hz, caused by the onset of neutrino free-streaming, as reflected in the tensor transfer function. Most importantly, the entire range of models discussed here should be accessible to future CMB polarization experiments [16, 17, 18] and space-based gravitational wave detectors, like the Big Bang Observer (BBO) [19].

Hence, we find that, contrary to some suggestions in the literature, all inflationary models are not created equal. The goals of inflation do not require going beyond models with minimal or near-minimal tuning ($Z_\eta \leq 1$). (For skeptical readers who may demur from this conclusion, we pose a challenge: construct a deterministic, complete inflationary model forced by fundamental physics into a parameter region with $Z_\eta \gg 1$.) Furthermore, the minimal and near-minimal models are the most powerfully predictive in the sense that they require the fewest tunings and make the highest number of successful predictions. Based on this analysis, both a red tilt with $n_s < 0.98$ and cosmic gravitational waves with $r \gtrsim 10^{-2}$ are expected and should be detected if inflation is right. A similar analysis should be applied to cyclic models [20], which must satisfy some conditions analogous to the inflationary conditions (and some not), to determine if the same range of n_s is favored.

[1] G. Efstathiou and K. J. Mack, JCAP **0505**, 008 (2005); M. B. Hoffman and M. S. Turner, Phys. Rev. D **64**, 023506 (2001); W. H. Kinney, Phys. Rev. D **66**, 083508 (2002); H. V. Peiris *et al.*, Astrophys. J. Suppl. **148**, 213 (2003).

[2] R. Trotta, astro-ph/0504022.

[3] D. Cirigliano, H. J. de Vega and N. G. Sanchez, Phys. Rev. D **71**, 103518 (2005); D. Boyanovsky, H. J. de Vega and N. G. Sanchez, Phys. Rev. D **73**, 023008 (2006).

[4] Although the text states $N_0 \approx 60$, we compute the precise value of N_0 model by model in our analysis. Also, we assume instantaneous reheating, but it is straightforward to show that assuming a low reheat temperature (or preheating) does not change our conclusions.

[5] A. Linde, Phys. Rev. D **49**, R748 (1994).

[6] D. Baumann, P. J. Steinhardt and N. Turok, in preparation (2006).

[7] P. J. Steinhardt, Mod. Phys. Lett. A **19**, 967 (2004).

[8] D. H. Lyth, Phys. Rev. Lett. **78**, 1861 (1997).

[9] J. E. Kim, H. P. Nilles and M. Peloso, JCAP **0501**, 005 (2005); S. Dimopoulos, S. Kachru, J. McGreevy and J. G. Wacker, hep-th/0507205; R. Easther and L. McAllister, hep-th/0512102; M. Kawasaki, M. Yamaguchi and T. Yanagida, Phys. Rev. Lett. **85**, 3572 (2000).

[10] L. A. Boyle and P. J. Steinhardt, astro-ph/0512014.

[11] M. S. Turner, Phys. Rev. D**55**, R435 (1997); M. S. Turner, M. White and J. E. Lidsey, Phys. Rev. D **48**, 4613 (1993).

[12] T. L. Smith, M. Kamionkowski and A. Cooray, Phys. Rev. D **73**, 023504 (2006).

[13] S. Bashinsky, astro-ph/0505502.

[14] S. Weinberg, Phys. Rev. D **69**, 023503 (2004).

[15] U. Seljak *et al.*, Phys. Rev. D **71**, 103515 (2005).

[16] The Planck satellite will improve on the WMAP limit: see Fig. 2 in C. Burigana, L. A. Popa, F. Finelli, R. Salvaterra, G. De Zotti, N. Mandolesi, astro-ph/0411415.

[17] L. Verde, H. Peiris and R. Jimenez, JCAP **0601**, 019 (2006).

[18] R. Weiss *et al.*, Task Force On Cosmic Microwave Research, www.science.doe.gov/hep/TFCRreport.pdf

[19] E. S. Phinney, private communication (2005); C. Ungarelli, P. Corasaniti, R. A. Mercer and A. Vecchio, Class. Quant. Grav. **22**, S955 (2005).

[20] P. J. Steinhardt and N. Turok, Science **296**, 1436 (2002).