Inflation versus Cyclic Predictions for Spectral Tilt

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We present a nearly model-independent estimate that yields the predictions of a class of simple inflationary and ekpyrotic/cyclic models for the spectral tilt of the primordial density inhomogeneities that enables us to compare the two scenarios. Remarkably, we find that the two produce an identical result, $n_s \approx 0.95$. For inflation, the same estimate predicts a ratio of tensor to scalar contributions to the low l multipoles of the microwave background anisotropy of $T/S \approx 20\%$; the tensor contribution is negligible for ekpyrotic/cyclic models, as shown in earlier papers.

The recent measurement of the cosmic microwave background (CMB) anisotropy by WMAP [1] is consistent with a primordial power spectrum of density fluctuations that is scale-invariant, gaussian and adiabatic. These characteristics coincide with the predictions of the simplest inflationary scenarios [2].

In this paper, we show that these are also predicted by the simplest ekpyrotic/cyclic scenarios [3, 4, 5]. We compare the density fluctuation spectra obtained in inflationary [2] and ekpyrotic/cyclic models by computing their predictions for an important, well-motivated class of simple models. We find surprisingly similar predictions for the spectral index of the scalar density fluctuations [6]. Both predict a red spectrum with index $n_s \approx 0.95$. For inflation, the same argument predicts a ratio of tensor to scalar contributions to the low l multipoles of roughly 20 percent. Our results for inflation are not new; the particular form of the argument presented here is a variant of the discussion in Ref. [7] and by V. Mukhanov [8] and gives a similar result to other estimates. But our result for the ekpyrotic/cyclic models and the similarity to the inflationary prediction is both new and unexpected.

For the ekpyrotic/cyclic models, scale-invariant fluctuations are generated during a period of slow contraction. The notion is that these imprint themselves as temperature fluctuations in the current expanding phase [9]. The validity of this idea has been debated [9, 10, 11], with different answers obtained depending on assumptions about the precise matching conditions at the bounce. Here we use the recent results of Tolley *et al.* [12], which treat the bounce as a collision of branes in five dimensions, derive a unique matching condition, and find a scale-invariant spectrum of temperature fluctuations after the bounce.

Both inflation and the ekpyrotic/cyclic models rely on the equation of state parameter w having a specific qualitative behavior throughout the period when fluctuations are generated, including the interval when fluctuations with wavelengths within the present horizon radius were produced (corresponding to the last $\mathcal{N} \approx 60$ e-folds in wavelength). For inflation, the condition on w is that $1 + w \ll 1$ and for ekpyrotic/cyclic models it is $w \gg 1$ [9, 13]. Correspondingly, the Hubble constant H is nearly constant during inflation and the four dimensional scale factor a is nearly constant during ekpyrosis. Since these conditions must be maintained for the duration of an epoch spanning many more than $\mathcal{N}$ e-folds, the simplest possibility is to suppose that w (and correspondingly H or a) change slowly and monotonically during that last $\mathcal{N}$ e-folds. More precisely, we take "simplest" to mean that (i) $dw/d\mathcal{N}$ is small, and $d^2w/d\mathcal{N}^2$ or $(dw/d\mathcal{N})^2$ negligible and (ii) in order for inflation (ekpyrosis) to end, H during inflation (or a during ekpyrosis) decays by a factor of order unity over the last $\mathcal{N}$ e-folds. Tilts or spectral features that differ from those presented here can only be produced by introducing by hand unnecessary rapid variations in w – unnecessary in the sense that they are not required for either model to give a successful account of the standard cosmology.

Note that our condition on the time-variation of w does not refer directly to any particular inflaton or cyclic scalar field potential. In fact, it does not assume that either scenario is driven by a scalar field at all. But, one might ask: how does our condition on the equation of state translate into a condition on an inflaton potential? The answer is simple: it means that the potential is characterized by a single dimensionful scale, typically H_I, the Hubble parameter during inflation. For example, for many models the effective potential is well characterized as $M^4 f(\phi/M)$, where ϕ is the inflaton field, $H_I \approx M^2/M_{Pl}$ where M_{Pl} is the Planck mass, and $f(x)$ is a smooth function which, when expanded in ϕ/M, has all dimensionless parameters of the same order [14]. In these cases, to produce inflationary models in which there are rapid changes in the equation of state in the last $\mathcal{N}$-folds, sharp features have to be introduced in the inflaton potential: bumps, wiggles, steep waterfalls, *etc.* But recall that the inflaton field is rolling very slowly throughout inflation, including the last $\mathcal{N}$ e-folds. Typically, ϕ rolls a short distance, $\Delta\phi \ll M$, during the last $\mathcal{N}$ e-folds. Hence, any sharp features must take place over a range $\delta\phi \ll \Delta\phi \ll M$, or equivalently, by introducing new fields or new mass scales much greater than M in the inflaton potential. For the purposes of comparing the inflationary and ekpyrotic/cyclic predictions, it makes most sense to consider the class with fewest parameters and simplest uniform behavior of the equation of state, a class which is also well-motivated in both models.

Recently, Gratton *et al.* [13] analyzed the conditions on the equation of state w required in order for quantum fluctuations in a single scalar field to produce nearly scale-invariant density perturbations, including models which (in the four dimensional effective description) bounce from a contracting to an expanding phase. Their analysis showed that there are only two cases which avoid extreme fine-tuning of initial conditions and/or the effective potential: $w \approx -1$ (inflation) and $w \gg 1$ (the ekpyrotic/cyclic scenario).

Following Gratton *et al.* [13], we shall discuss the production of long wavelength perturbations in the gauge invariant Newtonian potential Φ, which completely characterizes the density perturbation. Defining $u \equiv a\Phi/\phi'$ (henceforth, primes denote differentiation with respect to conformal time τ), then a Fourier mode of u with wavenumber k, u_k, obeys the differential equation

$$u_k'' + \left(k^2 - \frac{\beta(\tau)}{\tau^2}\right) u_k = 0\,, \tag{1}$$

with

$$\beta(\tau) \equiv \tau^2 H^2 a^2 \left\{ \bar{\epsilon} - \frac{(1-\bar{\epsilon}^2)}{2}\left(\frac{d\ln\bar{\epsilon}}{d\mathcal{N}}\right) + \frac{(1-\bar{\epsilon}^2)}{4}\left(\frac{d\ln\bar{\epsilon}}{d\mathcal{N}}\right)^2 - \frac{(1-\bar{\epsilon})^2}{2}\frac{d^2\ln\bar{\epsilon}}{d\mathcal{N}^2} \right\}, \tag{2}$$

where $H = a'/a^2$ is the Hubble parameter, and where $\bar{\epsilon}$ is related to the equation of state parameter w by

$$\bar{\epsilon} \equiv \frac{3}{2}(1+w)\,. \tag{3}$$

We have introduced the dimensionless time variable $\mathcal{N}$, defined by

$$\mathcal{N} \equiv \ln\left(\frac{a_{end}H_{end}}{aH}\right), \tag{4}$$

where the subscript "end" denotes that the quantity is to be evaluated at the end of the inflationary expansion phase or ekpyrotic contraction phase (corresponding to $w \gg 1$). Note that $\mathcal{N}$ measures the number of e-folds of modes which exit the horizon before the end of the inflationary or ekpyrotic phase. (N.B. $d\mathcal{N} = (\bar{\epsilon}-1)dN$ where $N = \ln a$ in Ref. [13].) Indeed, defining as usual the moment of horizon-crossing as $k_{\mathcal{N}} = aH$ for a given Fourier mode with comoving wavenumber $k_{\mathcal{N}}$, then

$$\mathcal{N} = \ln\left(\frac{k_{end}}{k_{\mathcal{N}}}\right), \tag{5}$$

where k_{end} is the last mode to be generated.

For nearly constant w (or constant $\bar{\epsilon}$), the unperturbed equations of motion have the approximate solution

$$a(\tau) \sim (-\tau)^{1/(\bar{\epsilon}-1)}\,, \qquad H = \frac{1}{(\bar{\epsilon}-1)a\tau}\,. \tag{6}$$

Substituting the second of these expressions in β, we find

$$\beta(\tau) \approx \frac{1}{(1-\bar{\epsilon})^2}\left\{ \bar{\epsilon} - \frac{(1-\bar{\epsilon}^2)}{2}\left(\frac{d\ln\bar{\epsilon}}{d\mathcal{N}}\right)\right\}, \tag{7}$$

where we have assumed that the higher-order derivative terms $d^2\ln\bar{\epsilon}/d\mathcal{N}^2$ and $(d\ln\bar{\epsilon}/d\mathcal{N})^2$ are much smaller than $d\ln\bar{\epsilon}/d\mathcal{N}$.

With the approximation that β is nearly constant for all modes of interest, Eq. (1) can be solved analytically, and the resulting deviation from scale invariance is simply given by the master equation

$$n_s - 1 \approx -2\beta \approx -\frac{2}{(1-\bar{\epsilon})^2}\left\{ \bar{\epsilon} - \frac{(1-\bar{\epsilon}^2)}{2}\left(\frac{d\ln\bar{\epsilon}}{d\mathcal{N}}\right)\right\}. \tag{8}$$

Inflation. Inflation is characterized by a period of superluminal expansion during which $w \approx -1$; that is, $\bar{\epsilon} \ll 1$. In this case, Eq. (8) reduces to

$$n_s - 1 \approx -2\bar{\epsilon} + \frac{d\ln\bar{\epsilon}}{d\mathcal{N}}\,, \tag{9}$$

as derived by Wang *et al.* [14].

The next step consists in rewriting the above in terms of $\mathcal{N}$ only. For this purpose, we need a relation between $\bar{\epsilon}$ and $\mathcal{N}$. During inflation, the Hubble parameter is nearly constant, but the "end" means that H begins to change significantly. So, if we are considering the last $\mathcal{N}$ e-folds, then, using Eqs. (6) and the definition of $\mathcal{N}$ (see Eq. (4)), it must be that H decays by a factor of order unity over those $\mathcal{N}$ e-folds or

$$\frac{H_{end}}{H} = \left(\frac{a}{a_{end}}\right)^{\bar{\epsilon}} \approx e^{-\bar{\epsilon}\mathcal{N}} \approx e^{-1}\,, \tag{10}$$

or

$$\bar{\epsilon} \approx \frac{1}{\mathcal{N}}\,. \tag{11}$$

Assuming that this relation holds approximately for all relevant modes, we may substitute in Eq. (9) and obtain

$$(n_s - 1)_{inf} \approx -\frac{2}{\mathcal{N}} - \frac{1}{\mathcal{N}} = -\frac{3}{\mathcal{N}}\,. \tag{12}$$

Note that, in this approximation, the two terms on the right hand side of Eq. (12) are both of order $1/\mathcal{N}$. Figuring that our approximation is good to order $1/\mathcal{N}$ or a few percent, the result is in agreement with the tilt predicted by simple inflationary models [15].

To obtain a numerical estimate of n_s, we may derive an approximate value for $\mathcal{N}$ from the observational constraint that the amplitude of the density perturbations, $\delta\rho/\rho$, be of order 10^{-5}. In the simplest inflationary models, $\delta\rho/\rho$ is given by [6]

$$\frac{\delta\rho}{\rho} \approx \left(\frac{T_r}{M_{Pl}}\right)^2 \bar{\epsilon}^{-1/2} \approx \left(\frac{T_r}{M_{Pl}}\right)^2 \mathcal{N}^{1/2} \sim 10^{-5}\,, \tag{13}$$

where T_r is the reheat temperature. On the scale of the observable universe today, $\mathcal{N}$ has the value (see Eq. (4))

$$\mathcal{N} = \ln\left(\frac{a_{end}H_{end}}{a_0 H_0}\right) \approx \ln\left(\frac{T_r}{T_0}\right), \tag{14}$$

where a_0, H_0 and T_0 are respectively the current values of the scale factor, Hubble parameter and (photon) temperature. For simplicity, we have assumed that $H_{end} \sim T_r^2/M_{Pl}$. Combining Eqs. (13) and (14), we obtain the constraint

$$e^{\mathcal{N}}\mathcal{N}^{1/4} \approx 10^{-5/2}\frac{M_{Pl}}{T_0}, \tag{15}$$

which implies $\mathcal{N} \approx 60$. It follows that $T_r \sim 10^{16}$ GeV.

If we substitute $\mathcal{N} \approx 60$ in Eq. (12), we obtain $n_s \approx 0.95$, within a percent or two of what is found for the simplest slow-roll and chaotic potentials [16, 17].

The prediction for the ratio of tensor to scalar contributions to the quadrupole of the CMB for a model with 70% dark energy and 30% matter is, then, [16, 17, 18]

$$T/S \approx 13.8\bar{\epsilon} \approx \frac{13.8}{\mathcal{N}} \approx 23\%, \tag{16}$$

which is very pleasing because it is in the range which is potentially detectable in the fluctuation spectrum and/or the CMB polarization in the near future [19]. (The WMAP collaboration [20] uses a different convention for T/S, defining $(T/S)_{WMAP}$ as the ratio of tensor to scalar amplitude of the primordial spectrum. The conversion factor to our T/S is $(T/S)_{WMAP} \approx 1.16\,(T/S)$.)

It is sometimes said that it is easy to construct models where T/S is very small, less than 1%, say. The argument is that the amplitude of tensor fluctuations is proportional to H^2, and a modest decrease in the energy scale for inflation reduces the tensor amplitude significantly. However, one must also consider Eq. (16) combined with Eq. (11). From Eq. (16), making T/S less than 1%, for instance, requires $\bar{\epsilon} < 10^{-3}$, which implies $\mathcal{N} > 1000$. Since we are interested in T/S at $\mathcal{N} \approx 60$, however, the only way to accommodate such a small $\bar{\epsilon}$ at $\mathcal{N} \approx 60$ is to have $\bar{\epsilon}$ make a rapid change at some point between $\mathcal{N} \approx 60$ and the end of inflation. This is precisely what is done in models which yield a small T/S ratio. (Restated in terms of the inflation potential $V(\phi)$, in order to have $T/S \approx 13.8\bar{\epsilon} \approx 28\ (d\ln V/d\phi)^2 \ll 1\%$, it must be that $d\ln V/d\phi \ll 0.02$, which is too small if inflation is to end in 60 e-folds unless one introduces a very rapid change in the slope during the last 60 e-folds.)

Ekpyrotic/cyclic models. The ekpyrotic phase is characterized by a period of slow contraction with $w \gg 1$; that is, $\bar{\epsilon} \gg 1$. In this case, Eq. (8) reduces to [13]

$$n_s - 1 \approx -\frac{2}{\bar{\epsilon}} - \frac{d\ln\bar{\epsilon}}{d\mathcal{N}}. \tag{17}$$

Notice that this relation can be transformed into the expression for inflation, Eq. (9), by replacing $\bar{\epsilon} \to 1/\bar{\epsilon}$. Note further that, for all cosmologies, the scale factor is $a \propto t^{1/\bar{\epsilon}} \propto H^{-1/\bar{\epsilon}}$, where t is proper time. Hence, inflation ($\bar{\epsilon} \ll 1$) has a rapidly varying and H nearly constant, whereas the ekpyrotic/cyclic model ($\bar{\epsilon} \gg 1$) has H varying and a nearly constant. This suggests an interesting duality between the inflationary and ekpyrotic/cyclic models that reflects itself in the final results.

If the scale factor $a(\tau)$ is nearly constant during the ekpyrotic (contraction) phase, then the phase ends when $a(\tau)$ begins to change significantly. In particular, the condition that the scale factor $a(\tau)$ decays by a factor of order unity during the last $\mathcal{N}$ e-folds reads

$$\frac{a_{end}}{a} = \left(\frac{aH}{a_{end}H_{end}}\right)^{1/(\bar{\epsilon}-1)} \approx e^{-\mathcal{N}/\bar{\epsilon}} \approx e^{-1} \tag{18}$$

(the analogue of Eq. (10) for inflation), which implies

$$\bar{\epsilon} \approx \mathcal{N} \tag{19}$$

(to be compared with Eq. (11) for inflation). Substituting this expression into Eq. (17), one obtains

$$(n_s - 1)_{ek} \approx -\frac{2}{\mathcal{N}} - \frac{1}{\mathcal{N}} = -\frac{3}{\mathcal{N}}. \tag{20}$$

This is the key relation for the ekpyrotic/cyclic models.

In the inflationary case, we estimated $\mathcal{N}$ by using the constraint on the amplitude of density perturbations, $\delta\rho/\rho \sim 10^{-5}$. For ekpyrotic and cyclic models, this constraint involves more parameters and is therefore not sufficient by itself to fix $\mathcal{N}$ [12, 21]. To estimate $\mathcal{N}$, we rewrite Eq. (4) as

$$\mathcal{N} \approx \ln\left(\frac{T_r}{T_0}\right) + \ln\left(\frac{a_{end}H_{end}}{a_r H_r}\right), \tag{21}$$

where the subscript r denotes the onset of the radiation-dominated phase. In inflation, we have $a_{end} \approx a_r$ and $H_{end} \approx H_r$. In the ekpyrotic/cyclic model, however, the end of ekpyrosis occurs during the contracting phase whereas the onset of radiation-domination is during the expanding phase. To estimate the ratio $a_{end}H_{end}/a_r H_r$, we note that, from approximately the end of ekpyrosis, through the bounce, and up to the onset of radiation-domination, the universe is dominated by scalar field kinetic energy, *i.e.*, $w \approx 1$ [4, 5]. From Eqs. (3) and (6), we find $a \approx (-\tau)^{1/2} \sim H^{-1/3}$, and therefore

$$\frac{a_{end}H_{end}}{a_r H_r} \approx \left(\frac{H_{end}M_{Pl}}{T_r^2}\right)^{2/3}. \tag{22}$$

Substituting in Eq. (21), we find

$$e^{\mathcal{N}} = \left(\frac{H_{end}^2}{T_r M_{Pl}}\right)^{1/3}\frac{M_{Pl}}{T_0}, \tag{23}$$

which is the analogue of Eq. (15).

The constraints on H_{end} and T_r in cyclic models are analyzed in Ref. [21] and the range of allowed values is presented. Central values are $T_r \approx 10^5$ GeV and $H_{end} \approx 10^5$

GeV, which, from Eq. (23), implies $\mathcal{N} \approx 60$. (By pushing parameters, $\mathcal{N}$ can be made to vary 20% or so one way or the other.) Substituting $\mathcal{N} = 60$ in the expression for the tilt gives $n_s \approx 0.95$, the same estimate obtained for inflation.

Conclusions. Remarkably, our estimates for the typical tilt in the inflationary and ekpyrotic/cyclic models are virtually identical. Both models predict a red spectrum, with spectral slope

$$n_s - 1 \approx -\frac{3}{\mathcal{N}}. \tag{24}$$

Furthermore, when adding observational constraints such as the COBE constraint that the amplitude of density fluctuations be of order 10^{-5}, both models yield $\mathcal{N} \approx 60$. This results in an identical prediction for the spectral tilt of $n_s \approx 0.95$. Furthermore, in both models, the time-variation of the equation of state contributes a correction of $\mathcal{O}(1)$ that reddens the spectrum. We have seen that this occurs because there is fascinating duality ($\bar{\epsilon} \to 1/\bar{\epsilon}$) between inflationary and ekpyrotic/cyclic conditions. This result was neither planned nor anticipated and suggests a deep connection between the expanding inflationary phase and the contracting ekpyrotic/cyclic phase. The key difference is that inflation also predicts a nearly scale-invariant spectrum of gravitational waves with a detectable amplitude. The predicted ratio of tensor to scalar CMB multipole moments at low l is $T/S \approx 20\%$. The tensor spectrum from cyclic models is strongly blue and exponentially small on cosmic scales [3, 22].

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