

Dynamical Selection of the Primordial Density Fluctuation Amplitude

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In inflationary models, the predicted amplitude of primordial density perturbations Q is much larger than the observed value ($\sim 10^{-5}$) for natural choices of parameters. To explain the requisite exponential fine-tuning, *anthropic selection* is often invoked, especially in cases where microphysics is expected to produce a complex energy landscape. By contrast, we find examples of ekpyrotic models based on heterotic M-theory for which *dynamical selection* naturally favors the observed value of Q .

According to our current understanding, all structure in the universe originated from primordial density fluctuations. Roughly speaking, primordial overdense regions acted as seeds that subsequently underwent gravitational collapse leading to stars and galaxies, while underdense regions emptied out to form the currently observed voids. As the COBE [1] and WMAP [2] satellite experiments have spectacularly demonstrated, these seeds were already present instants after the big bang. Inflationary and ekpyrotic models both provide mechanisms for generating these density fluctuations by amplifying quantum fluctuations - in the case of inflation, during a phase of accelerated expansion after the bang, and, in the case of ekpyrosis, during a phase of slow contraction before the bang. Both inflation and ekpyrosis can be modeled by a scalar field ϕ with potential $V(\phi)$; in the case of inflation the potential is required to be flat and positive, while for ekpyrosis it must be steep and negative. In both models, the shape of the potential determines the spectrum of the fluctuations as well as their amplitude. It is the latter that we focus on in this work.

In the case of inflation, it is well known that the predicted amplitude of primordial density perturbations Q is much larger than the observed value ($\sim 10^{-5}$) for natural choices of parameters. For example, for a single field undergoing slow-roll inflation, the number of e-folds of inflation remaining N and the density fluctuation amplitude Q as a function of ϕ are:

$$N \approx \frac{V}{V_{,\phi\phi}} \quad \text{and} \quad Q \approx \frac{H^2}{\dot{\phi}} \approx \frac{V^{3/2}}{V_{,\phi}}. \quad (1)$$

Here H denotes the Hubble rate and we work in reduced Planck units. To judge the fine-tuning, it is useful to introduce dimensionless parameters, such as $\alpha \equiv V_{,\phi\phi}^2/(V_{,\phi})^{4/3}$ and $\beta \equiv V_{,\phi\phi}^2/V$, which are $\mathcal{O}(1)$ if the potential is not fine-tuned. For example, for a power-law potential $V(\phi) = \lambda\phi^4$, $\alpha \approx 23\lambda^{2/3}$ and $\beta = 144\lambda$. The two relations in Eq. (1) combine to give

$$\alpha = Q^{4/3}/N^2. \quad (2)$$

This reproduces the familiar result that obtaining the observed value of Q , corresponding to $N \sim 60$, requires $\alpha = \mathcal{O}(10^{-10})$, fine-tuning to ten decimal places. Larger Q requires less fine-tuning, and so is parametrically favored. Alternatively, the same relation shows that, for

fixed tuning α , the number of e-folds of inflation grows with Q ; so larger Q produces universes that are exponentially greater in volume. Furthermore, larger Q means more structure which means more galaxies, stars and habitable spaces (up to a certain limiting value of Q perhaps of order $10^{-(2-3)}$). In other words, whether viewed parametrically, volumetrically or structure-wise, nothing favors the observed value of Q . For more complicated models, like hybrid inflation, there are more parameters, fields and dimensionless ratios to consider, but it remains the case that nothing favors the observed Q . *In short, bad inflation – producing lots of volume with the wrong cosmic properties – is more likely than good inflation!*

A longstanding hope has been to identify microphysics that fixes parameters uniquely that generate the observed Q and no other. To date, no accepted microphysics is known that has this property, although we do not exclude its possibility. (There are cases where tuning is technically natural – parameters maintain their tuned values after quantum corrections; however, technical naturalness does not, by itself, explain why Q has the observed value.) An alternative is that microphysics (such as string theory) leads to a complex energy landscape with a discretuum of possible values of inflationary parameters. The parameters that vary in this way are said to “scan.” In this case, the observed value of Q does not seem to be preferred in any way, and generically larger Q is favored. Anthropic selection is then invoked to resolve the bad inflation problem. An anthropic upper limit of $Q \lesssim 3 \cdot 10^{-4}$ has been suggested based on requiring planetary orbits, with a radius similar to Earth’s radius around the associated star, not to be disrupted within a timeframe of a billion years [3]. However, it is not clear how seriously to take this bound. It is based on requiring typical planets to have strong similarities to the Earth, which does not seem essential for life, and it does not weigh the fact that larger Q is much more likely (for the reasons described above). In fact, until Q was actually measured, values of Q that are an order of magnitude or two larger appeared compatible with our own observable universe. If we take this as a more conservative anthropic bound, then inflation favors these larger values and the anthropic selection does not explain the observed value.

In this paper, we consider the same issue for ekpyrotic/cyclic models in which density fluctuations are gen-

erated prior to the big bang and inflation is avoided. Although versions of these models can be constructed from conventional 4d scalar fields and potentials, here we will return to the setting that originally motivated the idea: heterotic M-theory [4]. In this case, the big crunch/big bang transition corresponds to the collision and bounce between branes (orbifold planes) along an extra spatial direction, and smoothing, flattening, and density perturbation generation occur as the two branes approach one another before the bounce. The colliding brane picture has been debated in the literature and is currently unproven. However, for the purposes of this study, we will assume it is viable and demonstrate a significant cosmological consequence; and to the extent to which the result is intriguing, it provides motivation for exploring this picture further to determine its validity.

We will show that, in the colliding brane picture, the correct value of Q results from non-anthropocentric, dynamical selection even if there is a complex landscape where parameters scan. More precisely, there is an upper bound on Q which depends explicitly on the value of Newton's constant and implicitly on the gauge coupling constants. Then, given the observed values for these physical constants, the upper bound on Q , which is dynamically selected, agrees precisely with its observed value.

In heterotic M-theory [5], six small extra dimensions are wrapped in a Calabi-Yau manifold, and our brane is separated from a parallel one by an additional, eleventh, extra dimension. In the 4d effective field theoretic description, the ekpyrotic phase is modeled by a steep, negative potential of the form $V = -V_0 e^{-\sqrt{2\epsilon}\phi}$, where $\epsilon \approx 10^2$ is called the fast-roll parameter. During this phase, our universe slowly contracts in the 4d effective description according to the scaling solution

$$a = (-\tau)^{1/\epsilon}, \quad \dot{\phi} = \sqrt{2}/(\sqrt{\epsilon}\tau), \quad V = -\epsilon H^2, \quad (3)$$

using a conformal time coordinate τ which is negative before the big bang, and approaches zero at the collision between branes (the big crunch/big bang transition). During the contraction phase, the scalar ϕ rolls down the potential to the maximal depth $-V_{ek}$, at which point the potential increases and approaches zero. At that minimum, which we denote by τ_k , the “kinetic” phase begins. During this phase, the energy density is dominated by the kinetic energy of ϕ , and the corresponding solution is

$$a = (-\tau)^{1/2}, \quad \dot{\phi} = \sqrt{3}/(\sqrt{2}\tau). \quad (4)$$

Matching the Hubble rates in the two phases, we find

$$-2\tau_k = \sqrt{\epsilon/V_{ek}}. \quad (5)$$

From the 11d braneworld point of view, the two parallel branes, which are at a distance d_{11} at the time τ_k , approach each other and collide at $\tau = 0$. Conventionally, the brane velocities relative to the center of mass are denoted by y_0 , and so, to leading order in y_0 , we have that

$$d_{11} = 2y_0|\tau_k|. \quad (6)$$

(The inter-brane velocity is actually $\tanh(2y_0)$, but our approximation is valid since the branes move at non-relativistic speeds in the regime of interest.) Combined with (5), we can obtain an expression for the potential minimum in terms of the collision velocity

$$V_{ek} = \epsilon y_0^2/d_{11}^2. \quad (7)$$

Since the fluctuation amplitude Q goes as the square root of the potential minimum, $Q \propto V_{ek}^{1/2}$, we have

$$Q \propto y_0. \quad (8)$$

This simple result is at the core of our argument. The brane collision velocity cannot be arbitrarily high - semi-classical studies of the brane collision have shown that radiation and matter are produced at the brane collision, in quantities increasing with y_0 [6]. Once the collision velocity reaches relativistic values, *i.e.* for $y_0 \gtrsim 0.1$, the matter density produced at the collision reaches the Hagedorn density, and the branes re-collapse rapidly under gravity. That is, the branes “stick” together for relativistic collision speeds, and no bounce or expanding universe follows. Thus, there is an upper limit on y_0 , and correspondingly, *an upper limit on the primordial density fluctuation amplitude Q !*

The exact numerical value of this upper limit Q_{max} will depend on the details of the ekpyrotic or cyclic model under study. We will illustrate our argument with one of the best-understood models to date, namely the heterotic M-theory colliding branes solution [7], which incorporates the entropic mechanism [8] for producing density perturbations (for a review, see [9]). In this model, the amplitude of the (nearly scale-invariant) density fluctuations is given by

$$Q^2 \approx \frac{\epsilon V_{ek}}{10^3} \approx \frac{y_0^2 \epsilon^2}{10^3 d_{11}^2}, \quad (9)$$

where we have used (7). As stated earlier, the upper limit on y_0 is about 0.1, while $\epsilon \approx 10^2$. So it remains to determine the appropriate value of d_{11} . In the colliding branes solution, the 11-dimensional distance between the branes was calculated in Eq. (3.38) of Ref. [7] to be

$$d = \begin{cases} |2y_0\tau| & \text{for } |\tau| \leq |\tau_{bntb}| \\ \frac{1}{2\alpha} \equiv d_{11} & \text{for } |\tau| \geq |\tau_{bntb}|. \end{cases} \quad (10)$$

A few clarifications are in order: the above expression verifies that close to the collision, the inter-brane velocity is indeed $2y_0$ (in [7], d was only evaluated in this regime). Before the time τ_{bntb} , the inter-brane distance is fixed (to leading order in y_0) at the constant value $d_{11} = 1/(2\alpha)$, where α parameterizes the 4-form flux threading the internal Calabi-Yau space. The time τ_{bntb} marks a special event, namely the moment when, from the higher-dimensional point of view, the negative-tension brane bounces off a naked singularity in the 11-dimensional bulk spacetime (the details are unimportant

for us here, but can be found in [7, 10]). In the effective theory, this corresponds to a bending of the scalar field trajectory [11], see Fig. 1. What is important is that, throughout the ekpyrotic phase, the inter-brane distance is also fixed at the value d_{11} , and this implies that, in a cyclic context, we can identify d_{11} with its current value. As first discussed by Hořava and Witten [12, 13], the volume of the internal space V_{CY} determines the gauge coupling constant of grand unification, while both V_{CY} and d_{11} determine Newton's constant G_N . This allows us to fix the value of d_{11} at the phenomenologically required value of (in reduced Planck units) [14]

$$d_{11} \approx 10^{3.5}. \quad (11)$$

We are finally in a position to evaluate our bound on the fluctuation amplitude Q : since by definition $|\tau_{bntb}| < |\tau_k|$, using (9) we obtain the upper limit

$$Q_{max} \approx 10^{-4}, \quad (12)$$

which shows that this model yields an upper bound on Q that is very close to the observed value. Moreover, we note that, since there must be some time between the time at which the potential minimum is reached and the time when the trajectory bends, a more realistic approximation would be $|\tau_k| \approx (2-3)|\tau_{bntb}|$; with this improved estimate, the maximal value is $Q_{max} \approx 10^{-4.5}$, which, considering the level of approximation we are working at, coincides with the observed value.

Hence, considering only the formation of large-scale structure, the bounce that produces the highest possible Q is preferred because it produces more galaxies, stars, etc. *By imposing the prerequisite that there be a bounce, this purely dynamical selection principle naturally picks out the observed amplitude of primordial density fluctuations.* One could argue that we were successful numerically because we imposed the constraint that Newton's constant G_N and the gauge couplings have the observed values. If these couplings also were to scan, then the preferred Q would be different from one landscape minimum to the next. This does not diminish the significance of our result, though. In this case, we amend our conclusion to read that the ekpyrotic model predicts a correlation on the landscape between G_N , the gauge couplings and Q : that is, for minima that have G_N and gauge couplings like those we observe, Q is predicted to be $\mathcal{O}(10^{-(4-5)})$.

In order to complete our argument, we will now show what other effects the change in V_{ek} causes to the ekpyrotic or cyclic universe, in particular in terms of volume and tuning, and that these changes do not alter our result. The radiation energy density at the bang is expected to be proportional to the energy of the collision,

$$H_r^2 \approx T_r^4 \propto y_0^2, \quad (13)$$

where H_r and T_r denote the Hubble rate and temperature at the onset of radiation domination. However, as can be inferred from Eq. (7), the energy of the collision is proportional to V_{ek} . Moreover, the expansion of the

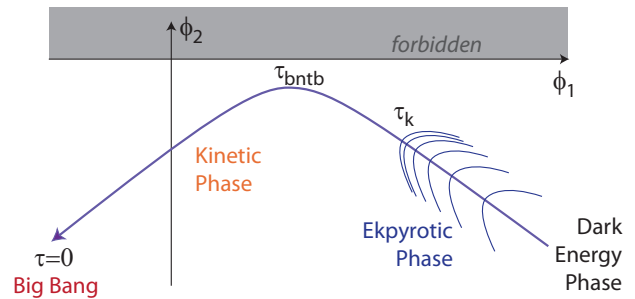


FIG. 1: The colliding branes solution, as seen in the scalar field space of the 4d effective theory. After the ekpyrotic phase comes to an end at τ_k , the kinetic phase begins. The trajectory bends at time τ_{bntb} before shooting off to $\phi_1, \phi_2 \rightarrow -\infty$, where the brane collision/big bang occurs at $\tau = 0$. After the big bang, the trajectory comes back along the same path, and a new cycle starts. In the figure, ϕ_1, ϕ_2 are related to the inter-brane distance and the volume of the Calabi-Yau space. Our scalar ϕ is tangential to the background trajectory, and is thus a function of ϕ_1 and ϕ_2 [15].

universe since the big bang is given by (T_r/T_0) , where T_0 denotes the current temperature of the cosmic microwave background. Thus, in the context of the ekpyrotic model, where we assume a one-time ekpyrotic phase followed by a brane collision/big bang, we can conclude that, by varying V_{ek} , the volume of the universe changes as $V_{ek}^{3/4}$. Therefore increasing V_{ek} , and thus increasing Q , implies that a larger universe is produced; that is, higher Q (but consistent with a bounce) is preferred volumetrically.

For the cyclic universe, the analysis is more involved. The main feature of the cyclic universe is that it provides a resolution of the problem of initial conditions in that it dynamically selects the right initial conditions cycle after cycle. However, in the two-field cyclic universe which we are discussing here, during the ekpyrotic phase there is an instability associated with the field direction transverse to ϕ [8, 16]. This instability has the consequence that only a small fraction of the universe makes it from one cycle to the next, but this is more than compensated for by the net expansion over the course of one cycle, if there is a sufficient amount of dark energy expansion [17, 18]. Then, overall, the volume of habitable space grows from cycle to cycle. Let us now look at the details:

The ekpyrotic phase lasts for a number $N_{ek} \equiv \frac{1}{2} \ln(V_{ek}/V_0)$ of e-folds of slow contraction (with $V_0 \approx T_0^4$ denoting the current dark energy scale). During this phase, the universe contracts by a negligible total amount, but a fraction $e^{-3N_{ek}}$ of the volume of the universe is thrown off the ideal cyclic trajectory due to the above-mentioned instability. These regions are likely to experience a fatal chaotic mixmaster big crunch and never bounce; they are irrelevant for galaxy formation.

The net increase in volume per cycle is given by [19]

$$\left(\frac{V_{ek}^{1/4}}{T_r}\right)^2 \left(\frac{T_r}{T_0}\right)^3 e^{3N_{de}}, \quad (14)$$

where the successive factors denote the expansion during the kinetic phase, radiation and matter domination and during the dark energy phase (lasting N_{de} e-folds). The scalar field ϕ reaches $-\infty$ at the big crunch/big bang transition [20], and rebounds in a matter of instants to the value near what it is today. To be precise, the field is stopped by the Hubble damping to radiation at $\phi_{stop} \sim \ln(y_0/H_r)$, before turning back. Note that, the bigger the collision velocity is, the further away from $-\infty$ the field rebounds; but the greater is the radiation damping, the quicker the field comes to a halt. From the relation (13) above, the net result is that ϕ_{stop} is unchanged. This also means that the number of e-folds of dark energy $N_{de} \approx e^{\sqrt{\epsilon}\phi_{stop}}/\epsilon$ is unchanged and independent of V_{ek} .

Putting these relations together, we conclude that the fractional increase F in habitable volume per cycle is given by $F = (T_r/T_0)e^{-2N_{ek}+3N_{de}}$. Since we are assuming that $T_r \propto V_{ek}^{1/4}$, this can be rewritten as

$$F = (e^{-\frac{1}{2}N_{ek}+N_{de}})^3. \quad (15)$$

Incidentally this shows that the requirement for the two-field cyclic universe to be sustainable is $N_{de} \geq \frac{1}{2}N_{ek}$. Since N_{de} is fixed, we have that

$$F \propto V_{ek}^{-3/4}. \quad (16)$$

Thus, increasing the depth of the potential actually produces a smaller universe. However, since an increase in V_{ek} also corresponds to an increase in $Q \propto V_{ek}^{1/2}$, this smaller universe will contain exponentially more structure. If we use Press-Schechter [21] as an approximation, then the probability for a region to have a density contrast δ is proportional to $(1/Q)e^{-\delta^2/Q^2}$. In our (region of the) universe, we have $Q^2 \approx 10^{-9}$. Hence, in the context

of the cyclic universe, decreasing the value of $V_{ek}^{1/4}$ by a factor of 10 produces a universe that is 10^3 times bigger but produces a value of Q that is 100 times smaller; the net effect is to decrease the probability of galaxies (and life) exponentially. So, as in the simple ekpyrotic case, the largest V_{ek} compatible with a bounce is preferred.

In sum, we have pointed out that the conventional inflationary picture suffers from the bad inflation problem: regions with Q larger than observed are preferred. They require less fine-tuning, produce more structure and, in most cases, generate more volume. In cases with a complex energy landscape in which inflationary parameters scan, the problem is particularly acute – our observed universe is highly unlikely. Anthropic selection reduces but does not resolve the problem. As an alternative, we have presented an example of an ekpyrotic/cyclic model where the observed value of Q is selected dynamically, even in cases where there is a complex energy landscape. Being able to do this when parameters scan is novel. This raises the interesting question, which we leave to future work, of what other predictions can be extracted from this framework. Note that we do not claim that the model presented here offers the unique explanation of the value of Q ; nor do we argue that dynamical selection with a complex energy landscape is impossible for inflationary models. Rather we want to provide an existence proof that a dynamical selection is possible and suggest that such an explanation should be added as an advantageous criterion in judging models of the early universe.

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