The Cosmic Gravitational-Wave Background in a Cyclic Universe

Latham A. Boyle[1], Paul J. Steinhardt[1], and Neil Turok[2]

[1]*Department of Physics, Princeton University, Princeton, New Jersey 08544*
[2]*Department of Applied Mathematics and Theoretical Physics,*
Centre for Mathematical Sciences, University of Cambridge,
Wilberforce Road, Cambridge CB3 OWA, United Kingdom
(Dated: July 2003)

Inflation predicts a primordial gravitational wave spectrum that is slightly "red," *i.e.* nearly scale-invariant with slowly increasing power at longer wavelengths. In this paper, we compute both the amplitude and spectral form of the primordial tensor spectrum predicted by cyclic/ekpyrotic models. The spectrum is exponentially suppressed compared to inflation on long wavelengths, and the strongest constraint emerges from the requirement that the energy density in gravitational waves should not exceed around 10 per cent of the energy density at the time of nucleosynthesis.

The recently-proposed cyclic model[1, 2] differs radically from standard inflationary cosmology [3, 4], while retaining the inflationary predictions of homogeneity, flatness, and nearly scale-invariant density perturbations. It has been suggested that the cosmic gravitational wave background provides the best experimental means for distinguishing the two models. Inflation predicts a nearly scale-invariant (slightly red) spectrum of primordial tensor perturbations, whereas the cyclic model predicts a blue spectrum.[1] The difference arises because inflation involves an early phase of hyper-rapid cosmic acceleration, whereas the cyclic model does not.

In this paper, we compute the gravitational wave spectrum for cyclic models to obtain both the normalization and spectral shape as a function of model parameters, improving upon earlier heuristic estimates. We find that the spectrum is strongly blue. The amplitude is too small to be observed by currently proposed detectors on all scales. Hence, the discovery of a stochastic background of gravitational waves would be evidence in favor of inflation, and would rule out the cyclic model.

Readers unfamiliar with the cyclic model may consult [5] for an informal tour, and [6] for a recent analysis of phenomenological constraints. Cyclic cosmology draws strongly on earlier ideas associated with the "ekpyrotic universe" scenario. [7, 8, 9] Briefly, the scenario can be described in terms of the periodic collision of orbifold planes moving in an extra spatial dimension, or, equivalently, in terms of a four-dimensional theory with an evolving (modulus) field ϕ rolling back and forth in an effective potential $V(\phi)$. The field theory description is the long wavelength approximation to the brane picture in which the potential represents the interbrane interaction and the modulus field determines the distance between branes. For the purposes of this paper, the field theoretic description is more useful.

The potential (Fig. 1) is small and positive for large ϕ, falling steeply negative at intermediate ϕ, and increasing again for negative ϕ. Each cycle consists of the following stages: (1) ϕ large and decreasing: the universe expands at an accelerated rate as $V(\phi) > 0$ acts as dark energy;
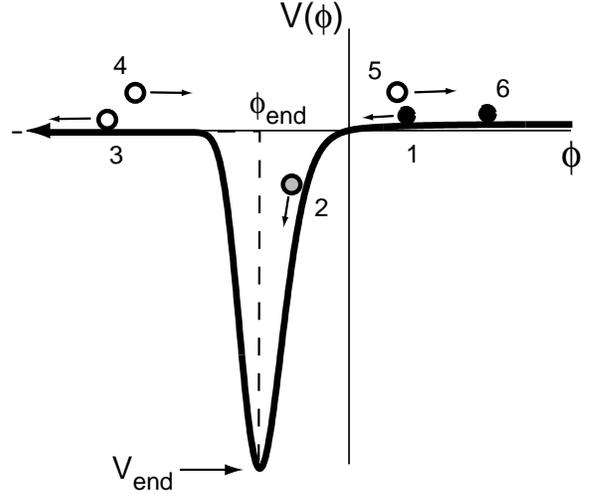

FIG. 1: Schematic of cyclic potential with numbers representing the stages described in the text. To the left of ϕ_{end}, where the scalar kinetic energy dominates, we approximate V with a Heaviside function, jumping to zero as shown by the dashed line.

(2) ϕ intermediate and decreasing: the universe is dominated by a combination of scalar kinetic and potential energy, leading to slow contraction and to the generation of fluctuations; (3) ϕ negative and decreasing (beginning at conformal time $\tau_{end} < 0$): the generation of fluctuations ends, ϕ rolls past ϕ_{end} and, in the four-dimensional description, the universe contracts rapidly, dominated by scalar field kinetic energy, to the bounce ($\tau = 0$) at which matter and radiation are generated; (4) ϕ increasing from minus infinity: the universe remains dominated by scalar field kinetic energy, which decreases rapidly compared to the radiation energy; (5) ϕ large and increasing (beginning at $\tau_r > 0$): the scalar field kinetic energy red-shifts to a negligible value and the universe begins the radiation dominated expanding phase; (6) ϕ large and nearly stationary: the universe undergoes the transitions to matter and dark energy domination, and the cycle begins anew.

We model the scalar field potential as:

$$V(\phi) = V_0(1 - e^{-c\phi/M_{pl}})\Theta(\phi - \phi_{end}) \qquad (1)$$

where M_{pl} is the reduced Planck mass and $\Theta(\phi)$ the Heaviside step function. A potential of this form, with an exponentially steep form, is required by the cyclic model in order to produce an acceptable spectrum of cosmological perturbations. [2, 6] Choosing $c = 10$ for example results in a scalar spectral index $n_s = .96$ which is compatible with current constraints. The Heaviside function marks the end of the steeply decreasing part of the potential; for $\phi < \phi_{end}$ the potential is small and the universe is dominated by scalar field kinetic energy.

Our calculation begins in the "ekpyrotic phase," stage (2), with the Einstein-frame scale factor contracting:

$$a(\tau) = a_{end}\left(\frac{\tau - \tau_{ek}}{\tau_{end} - \tau_{ek}}\right)^\alpha, \qquad \tau < \tau_{end}, \qquad (2)$$

where $\alpha \equiv 2/(c^2 - 2) \ll 1$ and $\tau_{ek} \equiv (1 - 2\alpha)\tau_{end}$, being the conformal time the potential would have diverged to minus infinity had the exponential form continued. At $\tau = \tau_{end}$, the ekpyrotic phase ends and the "contracting kinetic phase," stage (3), begins:

$$a(\tau) = \left(\frac{-\tau}{(1+\chi)\tau_r}\right)^{1/2}, \qquad \tau_{end} < \tau < 0. \qquad (3)$$

At $\tau = 0$, the universe bounces and the "expanding kinetic phase," stage (4), begins:

$$a(\tau) = \left(\frac{\tau}{\tau_r}\right)^{1/2}, \qquad 0 < \tau < \tau_r. \qquad (4)$$

Radiation is produced at the bounce, but is less than the scalar kinetic energy until, at $\tau = \tau_r$, the expanding kinetic phase ends, and standard radiation-dominated, matter-dominated, and dark-energy-dominated epochs ensue. The transition times, τ_r and τ_{end}, are given by

$$\tau_r = (\sqrt{2}H_r)^{-1}, \qquad \tau_{end} = -\tau_r/\Gamma, \qquad (5)$$

and

$$\Gamma \equiv \left|\frac{\tau_r}{\tau_{end}}\right| = \left[\frac{1}{1+\chi}\left(\frac{2\alpha}{1-2\alpha}\right)\left(\frac{V_{end}}{H_r^2 M_{pl}^2}\right)\right]^{1/3}, \qquad (6)$$

where $H_r \equiv H(\tau_r)$ is the Hubble constant at τ_r, $V_{end} = -V(\phi_{end})$ is the depth of the potential at its minimum, and $\chi \ll 1$ is a small positive constant that measures the amount of radiation created at the bounce. Note that $a(\tau)$ and $a'(\tau)$ are both continuous at the transition time $\tau = \tau_{end}$, and we have chosen to normalize $a(\tau)$ to unity at the start of radiation domination ($a(\tau_r) = 1$).

THE PRIMORDIAL SPECTRUM, $\Delta h(k, \tau_r)$

A quasi-stationary, isotropic, stochastic background of gravitational waves is characterized by the quantity $\Delta h(k, \tau)$, the rms dimensionless strain per unit logarithmic wavenumber at time τ (i.e. the $\delta L/L$ that would be measured by a detector with sensitivity band centered on mode k and bandwidth $\Delta k = k$). Accounting for both polarizations, it is given by $\Delta h(k, \tau) = k^{3/2}|h_k(\tau)|/\pi$, where the Fourier amplitude $h_k(\tau)$ satisfies

$$h_k'' + 2\frac{a'}{a}h_k' + k^2 h_k = 0 \qquad (7)$$

and the boundary condition that the solution approaches the Minkowski vacuum at short distances

$$h_k(\tau) \to \frac{e^{-ik\tau}}{a(\tau)M_{pl}\sqrt{2k}} \quad \text{as} \quad \tau \to -\infty. \qquad (8)$$

To solve equation (7), it is useful to define $f_k(\tau) \equiv a(\tau)h_k(\tau)$ and rewrite (7) as

$$(f_k)'' + (k^2 - \frac{a''}{a})f_k = 0. \qquad (9)$$

During the ekpyrotic phase, $a(\tau)$ is given by (2), and the general solution of (9) is

$$f_k(\tau) = \sqrt{y}\left(A_1(k)H_n^{(1)}(y) + A_2(k)H_n^{(2)}(y)\right), \qquad (10)$$

where $A_{1,2}(k)$ are arbitrary constants, $n \equiv \frac{1}{2} - \alpha$, $y \equiv -k(\tau - \tau_{ek})$, and $H_n^{(1,2)}$ are the Hankel functions. The boundary condition (8) implies

$$A_1(k) = \frac{1}{2}\sqrt{\frac{\pi}{k}}, \qquad A_2(k) = 0, \qquad (11)$$

where we have dropped a physically irrelevant phase. In the contracting kinetic phase, stage (4), $a(\tau)$ is given by (3), and the general solution of (9) is

$$f_k(\tau) = \sqrt{-k\tau}\left(B_1(k)H_0^{(1)}(-k\tau) + B_2(k)H_0^{(2)}(-k\tau)\right) \qquad (12)$$

where $B_{1,2}(k)$ are arbitrary constants. Then, continuity of h_k and h_k' at $\tau = \tau_{end}$ implies

$$B_{1,2}(k) = \mp\frac{i\pi}{4}\sqrt{\frac{\pi\alpha}{2k}}x_e\left[H_1^{(2,1)}(x_e)H_n^{(1)}(2\alpha x_e) + \right.$$
$$\left. + H_0^{(2,1)}(x_e)H_{n-1}^{(1)}(2\alpha x_e)\right] \qquad (13)$$

where $x_e \equiv k|\tau_{end}|$. Finally, in the expanding kinetic phase, $a(\tau)$ is given by (4), and the general solution of (9) is

$$f_k(\tau) = \sqrt{k\tau}\left(C_1(k)H_0^{(1)}(k\tau) + C_2(k)H_0^{(2)}(k\tau)\right). \qquad (14)$$

To fix $C_{1,2}(k)$, we need to match the solution across $\tau = 0$. At the level of quantum field theory in curved space-time, the choice is essentially unique[11], and amounts to analytically continuing the positive (negative) frequency part of $h_k \equiv f_k/a$ around the origin in the lower (upper) half of the complex τ-plane, so $H_0^{(1,2)}(-k\tau) \rightarrow -H_0^{(2,1)}(k\tau)$. This yields

$$C_{1,2}(k) = -\sqrt{1+\chi}\, B_{2,1}(k)\,. \tag{15}$$

The pre-factor arises because $a(\tau)$ differs by a factor of $\sqrt{1+\chi}$ between the kinetic contracting and expanding phases; see Eqs. (3) and (4). Combining our results, we arrive at the "primordial" dimensionless strain spectrum at the beginning of the radiation dominated epoch:

$$\Delta h(k,\tau_r) = \begin{array}{l} (k^2/\pi M_{pl})\,\sqrt{2(1+\chi)\tau_r} \\ \left| B_2(k)H_0^{(1)}(x_r) + B_1(k)H_0^{(2)}(x_r) \right| \end{array} \tag{16}$$

where $x_r \equiv k\tau_r$ and $k < k_{end}$. For $k > k_{end}$, the spectrum is cut off because these modes are not amplified and, instead, Eq. (16) converges to the result for a static Minkowski background. Recall, in the string theory context, we are describing the collision of two orbifold planes and in the vicinity of the collision the background space-time becomes a nearly-trivial, flat, compactified Milne geometry, which is locally Minkowski.[8] After the bounce, in the Einstein-frame description we use here, the universe is in an expanding phase where modes no longer exit the horizon. Super-horizon modes are frozen, and sub-horizon modes are well described by the adiabatic (WKB) approximation so that new gravitational waves are not generated.

THE PRESENT-DAY SPECTRUM, $\Delta h(k,\tau_0)$

To convert from the primordial spectrum to the present-day spectrum $\Delta h(k,\tau_0) \equiv T_h(k)\Delta h(k,\tau_r)$ we need to know the transfer function, $T_h(k)$. To approximate $T_h(k)$, note that $\Delta h(k,\tau)$ is roughly time-independent outside the horizon, and decays as a^{-1} once a mode re-enters the horizon. Therefore, the transfer function is $\sim 1/(1+z_r)$ for modes already inside the horizon at the onset of radiation domination τ_r, and $\sim 1/(1+z_k)$ for modes that entered at red shift z_k between τ_r and τ_0. Using the fact that $H \propto a^{-2}$ during radiation domination and $H \propto a^{-3/2}$ during matter domination (neglecting the change in g_*) we find

$$T(k) \approx \left(\frac{k_0}{k}\right)^2 \left[1 + \frac{k}{k_{eq}} + \frac{k^2}{k_{eq}k_r} \right] \tag{17}$$

where $k_0 \equiv a_0 H_0$, $k_{eq} \equiv a_{eq}H_{eq}$, $k_r \equiv a_r H_r$, and $k_{end} \equiv a_{end}|H_{end}|$ denote the modes on the horizon today (τ_0), at matter-radiation equality (τ_{eq}), at the start
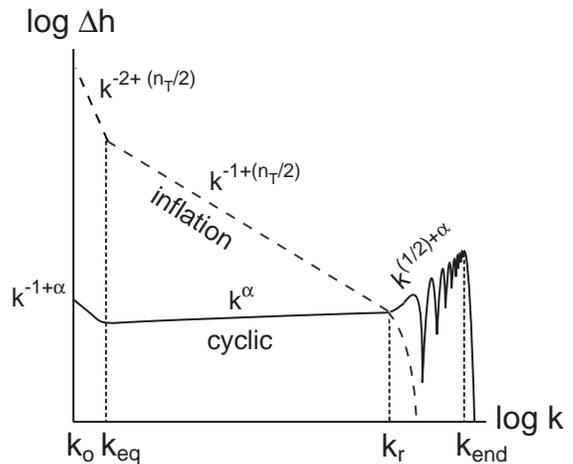

FIG. 2: A schematic comparison of the dimensionless strain observed today $\Delta h(k,\tau_0)$, as predicted by inflation and the cyclic model. Here n_T is the inflationary tensor spectral index (a small negative number), and $\alpha \ll 1$ in the cyclic model is a small positive number. k_r denotes the mode on the horizon at the start of radiation domination.

of radiation domination (τ_r), and at the end of the ekpyrotic phase (τ_{end}), respectively.

The gravitational wave spectrum can be divided into three regimes. There is a low frequency (LF) regime corresponding to long wavelength modes that re-enter after matter-radiation equality ($k < k_{eq}$), and a medium frequency (MF) regime consisting of modes which re-enter between equality and the onset of radiation domination ($k_{eq} < k < k_r$). (We ignore the recent dark energy dominated phase, which has negligible effect.) The spectrum for these two regimes is:

$$\Delta h \approx \frac{\Gamma^{\frac{1}{2}}k_0^2}{\pi M_{pl}H_r^\alpha} \begin{cases} k^{-1+\alpha} & (LF) \\ k^\alpha/k_{eq} & (MF) \end{cases} \tag{18}$$

Finally, modes which exit the horizon during the ekpyrotic phase (before τ_{end}), and re-enter during the expanding kinetic phase (after the bound but before τ_r) result in a high frequency (HF) band ($k_r < k < k_{end}$):

$$\Delta h \approx \left(\frac{\sqrt{2}}{\pi}\right)^{\frac{3}{2}} \frac{(\Gamma H_r)^{\frac{1}{2}-\alpha}k_0^2}{M_{pl}k_{eq}k_r} \left|\cos\left(k\tau_r - \frac{\pi}{4}\right)\right| k^{\frac{1}{2}+\alpha} \quad (HF) \tag{19}$$

The HF band runs over a range $k_{end}/k_r = \Gamma$, and this quantity is strongly constrained by the requirement that the scalar field cross the negative region of the potential before radiation domination begins, which requires that[6]

$$H_r \lesssim \frac{V_{end}^{\frac{1}{2}}}{M_{Pl}} \left(\frac{V_0}{V_{end}}\right)^{\sqrt{\frac{3}{2}}/c}, \tag{20}$$

where V_0 is today's value of the dark energy density. This equation, combined with (6), gives a lower bound on Γ, $\Gamma \gtrsim (V_{end}/V_0)\sqrt{2/3c^2}$. For example, for V_{end} around the GUT scale and $c = 10$, we find $\Gamma \geq 10^8$. Fig. 2 schematically depicts $\Delta h(k, \tau_0)$ in the cyclic scenario and compares it to the inflationary spectrum. [12]

Another useful quantity is $\Omega_{gw}(k, \tau_0)$, the gravitational wave energy per unit logarithmic wavenumber, in units of the critical density [10, 12]:

$$\Omega_{gw}(k, \tau_0) \equiv \frac{k}{\rho_{cr}}\frac{d\rho_{gw}}{dk} = \frac{1}{6}\left(\frac{k}{k_0}\right)^2 \Delta h(k, \tau_0)^2 . \quad (21)$$

In the cyclic model, $\Omega_{gw}(k, \tau_0)$ is very blue, with nearly all the gravitational wave energy concentrated at the high-frequency end of the distribution.

OBSERVATIONAL CONSTRAINTS AND DETECTABILITY

The strongest observational constraint on the gravitational spectrum in the cyclic model comes from the requirement that the successful predictions of big bang nucleosynthesis (BBN) not be affected, which requires

$$\int_{k_{BBN}}^{k_{end}} \Omega_{gw}(k, \tau_0)\frac{dk}{k} \lesssim \frac{0.1}{1 + z_{eq}} . \quad (22)$$

From the above equations, (19) and (21), and using $1 + z_{eq} \approx k_{eq}^2/k_0^2$, and $T_r \sim H_r^{\frac{1}{2}} M_{Pl}^{\frac{1}{2}}$ for the temperature at radiation domination, we obtain a total Ω in gravitational waves of $\sim (2\alpha V_{end}/T_r M_{Pl}^3)^{\frac{4}{3}}[36\pi^3(1 + z_{eq})]^{-1}$, which from (22) implies

$$T_r \gtrsim \frac{\alpha}{20}V_{end}M_{Pl}^{-3}, \quad (23)$$

where, for simplicity, we have ignored the factor which depends on the number of thermal degrees of freedom, which further weaken this bound.

The other observational constraints are much weaker. [13] From the CMB anisotropy, one infers $\Delta h(f \sim 10^{-18}\text{Hz}) \lesssim 10^{-5}$; from precision pulsar timing, $\Delta h(f \sim 10^{-8}\text{Hz}) \lesssim 10^{-14}$. Optimistic goals for LISA and advanced LIGO are strain sensitivities of $\Delta h(f \sim 10^{-4}\text{Hz}) \sim 10^{-20.5}$ and $\Delta h(f \sim 10^2\text{Hz}) \sim 10^{-24}$, respectively. Fig. 3 shows results for values of T_r and V_{end} consistent with all constraints on the cyclic model [6].

Even if the parameters are chosen to saturate the BBN constraint, the spectrum is still orders of magnitude below the sensitivity of anticipated instruments. In particular, it is hopeless to search for an imprint in the polarization of the CMB from cyclic model gravity waves because the predicted amplitude on large scales is so small. Hence, the detection of a stochastic gravitational wave imprint in the CMB polarization would be consistent with inflation and definitively rule out the cyclic model.
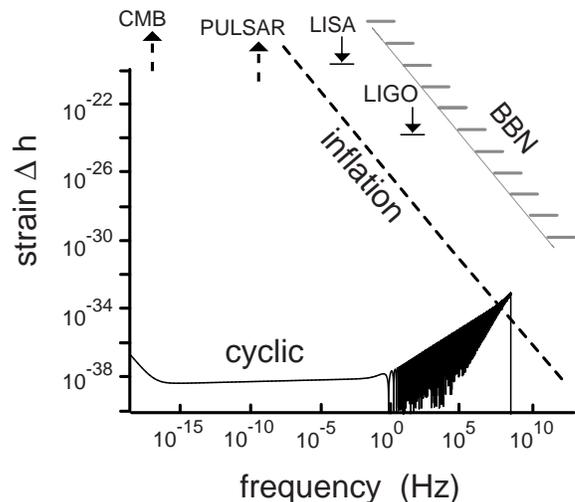

FIG. 3: The present-day dimensionless strain, $\Delta h(k, \tau_0)$, predicted by the cyclic model with $T_r = 10^7$ GeV and $V_{end}^{1/4} = 10^{14}$ GeV. These parameters yield a gravity wave density four orders of magnitude below the BBN bound. Some observational bounds and (optimistic) future strain sensitivities are indicated. The dashed arrows mean the empirical bounds lie well above range of Δh displayed here.

[1] P. J. Steinhardt and N. Turok, Science **296**, 1436-1439 (2002).
[2] P. J. Steinhardt and N. Turok, Phys. Rev. D **65**, 126003 (2002).
[3] A. H. Guth, Phys. Rev. D **23**, 347 (1981).
[4] A. D. Linde, Phys. Lett. B **108**, 389 (1982); A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982).
[5] P. J. Steinhardt and N. Turok, astro-ph/0204479.
[6] J. Khoury, P. J. Steinhardt, and N. Turok, hep-th/0307132, 2003.
[7] J. Khoury, B. A. Ovrut, P. J. Steinhardt, and N. Turok, Phys. Rev. D **65**, 123522 (2002).
[8] J. Khoury, B. A. Ovrut, N. Seiberg, P. J. Steinhardt, and N. Turok, Phys. Rev. D **65**, 086007 (2002).
[9] J. Khoury, B. A. Ovrut, P. J. Steinhardt, and N. Turok, Phys. Rev. D **66**, 046005 (2002).
[10] K. S. Thorne, in *300 Years of Gravitation*, edited by S. Hawking and W. Israel (Cambridge: Cambridge University Press, Cambridge, England, 1987), pp. 330–458.
[11] A. J. Tolley and N. Turok, Phys. Rev. D **66**, 106005 (2002); A.J. Tolley, N. Turok and P.J. Steinhardt, hep-th/0306109 (2003).
[12] M. S. Turner, Phys. Rev. D **55**, 435 (1997).
[13] B. Allen, gr-qc/9604033; K. S. Thorne, gr-qc/9506086.