Gravitational turbulent instability of AdS$_5$

Piotr Bizoń
A. Einstein Institute (Potsdam)
Jagiellonian University (Cracow)

joint work with Andrzej Rostworowski

Strings2014@Princeton, 23 June 2014
Anti-de Sitter spacetime in $d + 1$ dimensions

Manifold $\mathcal{M} = \{ t \in \mathbb{R}, x \in [0, \pi/2), \omega \in S^{d-1}\}$ with metric

$$g = \frac{\ell^2}{\cos^2 x} \left( -dt^2 + dx^2 + \sin^2 x d\omega_{S_{d-1}}^2 \right)$$

Spatial infinity $x = \pi/2$ is the timelike cylinder $\mathcal{I} = \mathbb{R} \times S^{d-1}$ with the boundary metric $ds_{\mathcal{I}}^2 = -dt^2 + d\Omega_{S_{d-1}}^2$

- Null geodesics get to infinity in finite time
- AdS is **not globally hyperbolic** - to make sense of evolution one has to prescribe boundary conditions at $\mathcal{I}$
- Asymptotically AdS spacetimes by definition have the same conformal boundary as AdS
Is AdS stable?

- By the positive energy theorem AdS space is the ground state among asymptotically AdS spacetimes (much as Minkowski space is the ground state among asymptotically flat spacetimes).

- Minkowski spacetime was proved to be asymptotically stable by Christodoulou and Klainerman (1993).

- Key difference between Minkowski and AdS: the mechanism of stability of Minkowski - **dissipation of energy by dispersion** - is absent in AdS (for no-flux boundary conditions \( \mathcal{I} \) acts as a mirror).

- The problem of stability of AdS has not been explored until recently; notable exceptions: proof of local well-posedness by Friedrich (1995), proof of rigidity of AdS (Anderson 2006).

- The problem seems tractable only in spherical symmetry so one needs to add matter to generate dynamics. Simple choice: a massless scalar field.
**AdS gravity with a spherically symmetric scalar field**

**Conjecture (B-Rostworowski 2011)**

\[ AdS_{d+1} \ (\text{for } d \geq 3) \text{ is unstable against black hole formation under arbitrarily small scalar perturbations} \]

Heuristic picture (supported by the nonlinear perturbation analysis and numerical evidence): due to resonant interactions between harmonics the energy is transferred from low to high frequencies. The concentration of energy on finer and finer scales eventually leads to the formation of a horizon (**strongly turbulent instability**).

- The turbulent instability is absent for some perturbations, in particular there is good evidence for the existence of **stable time-periodic solutions** (Maliborski-Rostworowski 2013)
- In $2+1$ dimensions there is a mass gap between AdS$_3$ and the lightest BTZ black hole. Small perturbations of AdS$_3$ remain smooth for all times but their radius of analyticity shrinks to zero as $t \to \infty$ (**weakly turbulent instability**) (B-Jałmużna 2013)
Other models

- Due to the computational limitations the numerical analysis of stability of AdS so far has been restricted to the $1 + 1$ dimensional setting (spherical symmetry).

- Which features of spherical collapse in the Einstein-scalar-AdS system are model-dependent and which ones hold in general?

- Other matter models: scalar field with $m^2 < 0$, Yang-Mills (allows for different boundary conditions and admits many static solutions)

- The vacuum case seems most interesting. The analysis of weak perturbations of AdS is very similar to the scalar field case (Dias-Horowitz-Santos 2012), however long-time numerical simulations without a symmetry reduction appear challenging

- A partial way around: one can evade Birkhoff’s theorem in five and higher odd spacetime dimensions
How to bypass Birkhoff in five dimensions

- Odd-dimensional spheres admit non-round homogeneous metrics
- Homogeneous metric on $S^3$

$$g_{S^3} = e^{2B} \sigma_1^2 + e^{2C} \sigma_2^2 + e^{2D} \sigma_3^2,$$

where $\sigma_k$ are left-invariant one-forms on $SU(2)$

$$\sigma_1 + i \sigma_2 = e^{i\psi} (\cos \theta \, d\phi + i \, d\theta), \quad \sigma_3 = d\psi - \sin \theta \, d\phi.$$

- $B = C = D$: round metric with $SO(4)$ symmetry
- $B \neq C \neq D$: anisotropic metric with $SU(2)$ symmetry (squashed $S^3$)

(B-Chmaj-Schmidt 2005): use $g_{S^3}$ as an angular part of the five dimensional metric (cohomogeneity-two triaxial Bianchi IX ansatz)

$$ds^2 = -Ae^{-2\delta} dt^2 + A^{-1} dr^2 + \frac{1}{4} r^2 \left( e^{2B} \sigma_1^2 + e^{2C} \sigma_2^2 + e^{-2(B+C)} \sigma_3^2 \right),$$

where $A, \delta, B, C$ are functions of $(t, r)$. The biaxial case: $B = C$. 
Cohomogeneity-two biaxial Bianchi IX ansatz in AdS

\[ ds^2 = \frac{\ell^2}{\cos^2 x} \left( -A e^{-2\delta} dt^2 + A^{-1} dx^2 + \frac{1}{4} \sin^2 x \left( e^{2B} (\sigma_1^2 + \sigma_2^2) + e^{-4B} \sigma_3^2 \right) \right), \]

where \( A, \delta, B \) are functions of \((t,x)\). Inserting this ansatz into the vacuum Einstein equations with \( \Lambda = -6/\ell^2 \) we get a hyperbolic-elliptic system

\[
\begin{align*}
(A^{-1} e^\delta \dot{B})' &= \frac{1}{\tan^3 x} \left( \tan^3 x A e^{-\delta} B' \right)' - \frac{4e^{-\delta}}{3 \sin^2 x} \left( e^{-2B} - e^{-8B} \right), \\
A' &= 4 \tan x \left( 1 - A \right) - 2 \sin x \cos x \left( A B'^2 + A^{-1} e^{2\delta} \dot{B}^2 \right) + \frac{2(4e^{-2B} - e^{-8B} - 3A)}{3 \tan x}, \\
\delta' &= -2 \sin x \cos x \left( B'^2 + A^{-2} e^{2\delta} \dot{B}^2 \right).
\end{align*}
\]

- We solve this system for smooth initial data \( B(0,x), \dot{B}(0,x) \) with finite mass \( M = \lim_{x \to \pi/2} \sin^2 x \sec^2 x (1 - A) \)
- Asymptotic behavior near infinity \((x = \pi/2)\)

\[ B(t,x) \sim b_\infty(t)(\pi/2 - x)^4, \quad \delta(t,x) \sim \delta_\infty(t), \quad 1 - A(t,x) \sim M(\pi/2 - x)^4 \]
Spectral properties

- Linearized equation:
  \[ \ddot{B} + LB = 0, \quad L = -\frac{1}{\tan^3 x} \partial_x \left( \tan^3 x \partial_x \right) + \frac{8}{\sin^2 x} \]

  The operator \( L \) is essentially self-adjoint on \( L^2([0, \pi/2), \tan^3 x dx) \).

- The eigenvalues and eigenfunctions of \( L \) are \((k = 0, 1, \ldots)\)
  \[ \omega_k^2 = (6 + 2k)^2, \quad e_k(x) = d_k \sin^2 x \cos^4 x \, _2F_1(-k, 6 + k, 4; \sin^2 x), \]
  where \( d_k \) is the normalization factor ensuring that \((e_j, e_k) = \delta_{jk}\).

- Using the generalized Fourier series \( B(t, x) = \sum_k b_j(t)e_k(x) \) we express the linearized energy as the Parseval sum
  \[ E = \int_0^{\pi/2} \left( \dot{B}^2 + B'^2 + \frac{8}{\sin^2 x} B^2 \right) \tan^3 x \, dx = \sum_k E_k, \]
  where \( E_k = \dot{b}_k^2 + \omega_k^2 b_k^2 \) is the energy of the \( k \)-th mode.
Blowup of the Kretschmann scalar

\[ R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} (t, 0) = 40 + 864 B''(t, 0)^2 \]
Key evidence for instability

Conjecture (B-Rostworowski 2014)

$AdS_5$ is unstable against black hole formation under arbitrarily small gravitational perturbations
Spectrum of energy

Universal power–law exponent $\alpha \approx -1.67$ (-5/3?)
Conclusions

- Dynamics of asymptotically AdS spacetimes is an interesting meeting point of basic problems in general relativity, PDE theory, AdS/CFT, and theory of turbulence. Understanding of these connections is at its infancy.

- Some open problems:
  - Turbulent instability is absent for some initial data. How big are these stability islands on the turbulent ocean?
  - Is the fully resonant linear spectrum necessary for the turbulent instability? (Dias, Horowitz, Marolf, Santos 2012).
  - Energy cascade has the power-law spectrum $E_k \sim k^\alpha$ with a universal exponent $\alpha$. What determines $\alpha$?
  - What happens outside spherical symmetry? It is not clear if the natural candidate for the endstate of instability - Kerr-AdS black hole - is stable itself (Holzegel-Smulevici 2013).
  - What are the implications of all that for the AdS/CFT?