A Proof of the Covariant Entropy Bound


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The World as a Hologram

- **The Covariant Entropy Bound** is a relation between information and geometry.  
  - RB 1999

- Motivated by holographic principle
  - Bekenstein 1972; Hawking 1974
  - ’t Hooft 1993; Susskind 1995; Susskind and Fischler 1998

- Conjectured to hold in arbitrary spacetimes, including cosmology.

- The entropy on a light-sheet is bounded by the difference between its initial and final area in Planck units.

- If correct, origin must lie in quantum gravity.
In this talk I will present a proof, in the regime where gravity is weak \((G\hbar \rightarrow 0)\).

Though this regime is limited, the proof is interesting.

No need to assume any relation between the entropy and energy of quantum states, beyond what quantum field theory already supplies.

This suggests that quantum gravity determines not only classical gravity, but also nongravitational physics, as a unified theory should.
Covariant Entropy Bound

Entropy $\Delta S$

Modular Energy $\Delta K$

Area Loss $\Delta A$
Surface-orthogonal light-rays

Any 2D spatial surface $B$ bounds four (2+1D) null hypersurfaces

- Each is generated by a congruence of null geodesics ("light-rays") $\perp B$
Out of the 4 orthogonal directions, usually at least 2 will initially be nonexpanding.

The corresponding null hypersurfaces are called light-sheets.
The Nonexpansion Condition

\[ \theta = \frac{dA/d\lambda}{A} \]

Demand

\[ \theta \leq 0 \iff \text{nonexpansion everywhere on the light-sheet.} \]
In an arbitrary spacetime, choose an arbitrary two-dimensional surface $B$ of area $A$. Pick any light-sheet of $B$. Then $S \leq A/4G\hbar$, where $S$ is the entropy on the light-sheet.
Example: Closed Universe

The light-sheets are directed towards the “small” interior, avoiding an obvious contradiction.
Generalized Covariant Entropy Bound

If the light-sheet is terminated at finite cross-sectional area $A'$, then the covariant bound can be strengthened:

$$ S \leq \frac{A - A'}{4G\hbar} $$

Flanagan, Marolf & Wald, 1999
Generalized Covariant Entropy Bound

\[ S \leq \frac{\Delta A}{4G\hbar} \]

For a given matter system, the tightest bound is obtained by choosing a nearby surface with initially vanishing expansion.

Bending of light implies

\[ A - A' \equiv \Delta A \propto G\hbar \]

Hence, the bound remains nontrivial in the weak-gravity regime \((G\hbar \to 0)\).
Covariant Entropy Bound

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How is the entropy defined?

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- In cosmology, and for well-isolated systems: usual, “intuitive” entropy. But more generally?

- Quantum systems are not sharply localized. Under what conditions can we consider a matter system to “fit” on $L$?

- The vacuum, restricted to $L$, contributes a divergent entropy. What is the justification for ignoring this piece?

In the $G\hbar \to 0$ limit, a sharp definition of $S$ is possible.
Vacuum-subtracted Entropy

Consider an arbitrary state $\rho_{\text{global}}$. In the absence of gravity, $G = 0$, the geometry is independent of the state. We can restrict both $\rho_{\text{global}}$ and the vacuum $|0\rangle$ to a subregion $V$:

$$
\rho \equiv \text{tr}_V \rho_{\text{global}} \\
\rho_0 \equiv \text{tr}_V |0\rangle\langle 0| 
$$

The von Neumann entropy of each reduced state diverges like $A/\epsilon^2$, where $A$ is the boundary area of $V$, and $\epsilon$ is a cutoff. However, the difference is finite as $\epsilon \to 0$:

$$
\Delta S \equiv S(\rho) - S(\rho_0) 
$$

Marolf, Minic & Ross 2003, Casini 2008
Covariant Entropy Bound

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Relative Entropy

Given any two states, the (asymmetric!) relative entropy

\[ S(\rho | \rho_0) = -\text{tr} \rho \log \rho_0 - S(\rho) \]

satisfies positivity and monotonicity: under restriction of \( \rho \) and \( \rho_0 \) to a subalgebra (e.g., a subset of \( \mathcal{V} \)), the relative entropy cannot increase.

Lindblad 1975
Modular Hamiltonian

Definition: Let $\rho_0$ be the vacuum state, restricted to some region $V$. Then the modular Hamiltonian, $K$, is defined up to a constant by

$$\rho_0 \equiv \frac{e^{-K}}{\text{tr } e^{-K}}.$$

The modular energy is defined as

$$\Delta K \equiv \text{tr } K \rho - \text{tr } K \rho_0.$$
A Central Result

Positivity of the relative entropy implies immediately that

\[ \Delta S \leq \Delta K. \]

To complete the proof, we must compute \( \Delta K \) and show that

\[ \Delta K \leq \frac{\Delta A}{4G\hbar}. \]
In finite spatial volumes, the modular Hamiltonian $K$ is nonlocal. But we consider a portion of a null plane in Minkowski:

$$x^- \equiv t - x = 0; \quad x^+ \equiv t + x; 0 < x^+ < 1.$$

In this case, $K$ simplifies dramatically.
Free Case

- The vacuum on the null plane factorizes over its null generators.
- The vacuum on each generator is invariant under a special conformal symmetry. Wall (2011)

Thus, we may obtain the modular Hamiltonian by application of an inversion, $x^+ \rightarrow 1/x^+$, to the (known) Rindler Hamiltonian on $x^+ \in (1, \infty)$. We find

$$K = \frac{2\pi}{\hbar} \int d^2 x^\perp \int_0^1 dx^+ \ g(x^+) \ T_{++}$$

with

$$g(x^+) = x^+(1 - x^+) \ .$$
In this case, it is not possible to define $\Delta S$ and $K$ directly on the light-sheet. Instead, consider the null limit of a spatial slab:

(a) (c) (b)
We cannot compute $\Delta K$ on the spatial slab.
However, it is possible to constrain the form of $\Delta S$ by analytically continuing the Rényi entropies,

$$S_n = (1 - n)^{-1} \log \text{tr} \rho^n,$$

to $n = 1$. 
Interacting Case

The Renyi entropies can be computed using the replica trick, as the expectation value of a pair of defect operators inserted at the boundaries of the slab. In the null limit, this becomes a null OPE to which only operators of twist \( d-2 \) contribute. The only such operator in the interacting case is the stress tensor, and it can contribute only in one copy of the field theory.

This implies

\[
\Delta S = \frac{2\pi}{\hbar} \int d^2 x^\perp \int_0^1 dx^+ g(x^+) \ T_{++}.
\]
Interacting Case

Because $\Delta S$ is the expectation value of a linear operator, it follows that

$$\Delta S = \Delta K$$

for all states.

Blanco, Casini, Hung, and Myers 2013

This is possible because the operator algebra is infinite-dimensional; yet any given operator is eliminated from the algebra in the null limit.
Interacting Case

We thus have

\[ \Delta K = \frac{2\pi}{\hbar} \int d^2 x^\perp \int_0^1 dx^+ g(x^+) \ T_{++}. \]

Known properties of the modular Hamiltonian of a region and its complement further constrain the form of \( g(x^+) \):

\( g(0) = 0, \ g'(0) = 1, \ g(x^+) = g(1 - x^+), \) and \( |g'| \leq 1. \)

I will now show that these properties imply

\[ \Delta K \leq \Delta A/4G\hbar, \]

which completes the proof.
Covariant Entropy Bound

Entropy $\Delta S$

Modular Energy $\Delta K$

Area Loss $\Delta A$
Integrating the Raychaudhuri equation twice, one finds

\[ \Delta A = -\int_{0}^{1} dx^{+} \theta(x^{+}) = -\theta_{0} + 8\pi G \int_{0}^{1} dx^{+} (1 - x^{+}) T_{++}. \]

at leading order in \( G \).
Area Loss in the Weak Gravity Limit

Integrating the Raychaudhuri equation twice, one finds

$$\Delta A = -\int_{0}^{1} dx^+ \theta(x^+) = -\theta_0 + 8\pi G \int_{0}^{1} dx^+ (1 - x^+) T_{++}.$$

at leading order in $G$. Compare to $\Delta K$:

$$\Delta K = \frac{2\pi}{\hbar} \int_{0}^{1} dx^+ g(x^+) T_{++}.$$

Since $\theta_0 \leq 0$ and $g(x^+) \leq (1 + x_+)$, we have $\Delta K \leq \Delta A/4G\hbar$. 
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at leading order in \( G \). Compare to \( \Delta K \):

\[ \Delta K = \frac{2\pi}{\hbar} \int_0^1 dx^+ g(x^+) T_{++}. \]

Since \( \theta_0 \leq 0 \) and \( g(x^+) \leq (1 + x_+) \), we have \( \Delta K \leq \Delta A/4G\hbar \)

if we assume the Null Energy Condition, \( T_{++} \geq 0 \).
It is easy to find quantum states for which $T_{++} < 0$.

Explicit examples can be found for which $\Delta S > \Delta A/4G\hbar$, if $\theta_0 = 0$.

Perhaps the Covariant Entropy Bound must be modified if the NEC is violated?

E.g., evaporating black holes

Surprisingly, we can prove $S \leq (A - A')/4$ without assuming the NEC.
Negative Energy Constrains $\theta_0$

- If the null energy condition holds, $\theta_0 = 0$ is the “toughest” choice for testing the Entropy Bound.
- However, if the NEC is violated, then $\theta_0 = 0$ does not guarantee that the nonexpansion condition holds everywhere.
- To have a valid light-sheet, we must require that
  \[
  0 \geq \theta(x^+) = \theta_0 + 8\pi G \int_{x^+}^1 d\hat{x}^+ \, T_{++}(\hat{x}^+) ,
  \]
  holds for all $x^+ \in [0, 1]$.
- This can be accomplished in any state.
- But the light-sheet may have to contract initially:
  \[
  \theta_0 \sim O(G\hbar) < 0 .
  \]
Proof of $\Delta K \leq \Delta A/4G\hbar$

Let $F(x^+) = x^+ + g(x^+)$. The properties of $g$ imply $F' \geq 0$, $F(0) = 0$, $F(1) = 1$.

By nonexpansion, we have $0 \geq \int_0^1 F' \theta dx^+$, and thus

$$\theta_0 \leq 8\pi G \int dx^+ [1 - F(x^+)] T_{++}.$$  \hspace{1cm} (1)

For the area loss, we found

$$\Delta A = -\int_0^1 dx^+ \theta(x^+) = -\theta_0 + 8\pi G \int_0^1 dx^+ (1 - x^+) T_{++}.$$  \hspace{1cm} (2)

Combining both equations, we obtain

$$\frac{\Delta A}{4G\hbar} \geq \frac{2\pi}{\hbar} \int_0^1 dx^+ g(x^+) T_{++} = \Delta K.$$  \hspace{1cm} (3)
In all cases where we can compute $g$ explicitly, we find that it is concave:

$$g'' \leq 0$$

This property implies the stronger result of monotonicity:

As the size of the null interval is increased, $\Delta S - \Delta A/4G\hbar$ is nondecreasing.

No general proof yet.
The Covariant Entropy Bound applies to any null hypersurface with $\theta \leq 0$ everywhere. It constrains the vacuum subtracted entropy on a finite null slab.

The GSL applies only to causal horizons, but does not require $\theta \leq 0$. It constrains the entropy difference between two nested semi-infinite null regions.

Limited proofs exist for both, but is there a more direct relation?