Singularities and Gauge Theory Phases

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With
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Other teams

- Kumar, Park, Taylor
- Grimm, Hayashi
- Krause, Mayrhofer, Weigand
- Hayashi, Lawrie, Morisson, Schafer-Nameki
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Love to my other collaborators:
Paolo Aluffi
Patrick Jefferson
Michele Del Zotto, Jonathan Heckman, Cumrun Vafa
At the menu:

- Geometry
- Gauge theories
- Representation theory
- Singularities
- Resolutions of singularities

It also comes as a combo:

...an elliptic fibration
# Tale of two worlds

<table>
<thead>
<tr>
<th>Gauge theories</th>
<th>Elliptic fibrations</th>
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<tr>
<td>Gauge algebra</td>
<td>Codimension one singularities</td>
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<td>Representations</td>
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<td>Yukawa</td>
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<td>Coulomb phases</td>
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<td>Walls</td>
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<td>Phase transitions</td>
<td>Flops</td>
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5D supersymmetric gauge theories with 8 supercharges

Matter content:

<table>
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<tr>
<th>Gravity multiplet</th>
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<tr>
<td>Vector multiplets</td>
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<tr>
<td>Hypermultiplets</td>
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<tr>
<td>Tensor multiplets</td>
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</table>
5D supersymmetric gauge theories with 8 supercharges

Geometry:

<table>
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<th>Vector multiplets</th>
<th>Very Special Geometry</th>
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<td>Hypermultiplets</td>
<td>Quaternionic-Kähler</td>
</tr>
<tr>
<td></td>
<td>Hyperkähler</td>
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</tbody>
</table>
Coulomb branch of 5D gauge theory

- Vector multiplets -> Weyl chamber
- Massless Hypers at singularities -> sub-chamber structure

\[
\mathcal{F}(\phi) = \frac{1}{2} m_0 \text{Tr}(\phi^2) + \frac{c_{cl}}{6} \text{Tr}(\phi^3) + \frac{1}{12} \left( \sum_{\alpha: \text{roots}} |(\phi, \alpha)|^3 - \sum_{w: \text{weights}} |(\phi, w)|^3 \right).
\]
Incidence geometry of a representation R

- $g$ is a Lie algebra with Cartan sub-algebra $h$
- Roots of $g$ define Weyl chambers
- $R$: a representation of $g$
- Weights of $R$ refine the Weyl chambers by adding a sub-chamber structure
Elliptic fibrations

- Elliptic curves are some of the oldest but yet most prominent objects across mathematics
  - Number theory
  - Algebraic geometry
  - Cryptography
  - Geometric design
  - Physics
Elliptic fibrations

Weierstrass model:

\[ Y : \quad y^2z + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2 + a_6z^3 \]
Singular fibers

- Kodaira
- Weierstrass model
- Néron models
- Tate’s algorithm
- Miranda models (collisions of singularities)
- Szydlo (generalization of Miranda’s model and Tate’s algorithm)
Singular fibers

- Kodaira classification (for elliptic surfaces)
Collision of singularities
(Miranda models)

\[ J = \infty : \quad I_n + I_m \quad \rightarrow \quad I_{n+m} \]
\[ I_{2n} + I_m^* \quad \rightarrow \quad I_{n+m}^* \]

\[ I_{2n+1} + I_m^* \quad \rightarrow \quad I_{n+m+1}^* \]

\[ J = 0 : \quad II + IV \quad \rightarrow \quad 1 \quad 2 \]
\[ II + I_0^* \quad \rightarrow \quad 1 \quad 2 \quad 3 \]
\[ IV + I_0^* \quad \rightarrow \quad 1 \quad 2 \quad 3 \quad 4 \quad 2 \]
\[ II + IV^* \quad \rightarrow \quad 1 \quad 2 \quad 4 \quad 2 \]

\[ J = 1 : \quad III + I_0^* \quad \rightarrow \quad 1 \quad 2 \quad 3 \quad 2 \quad 1 \]

( a total of \( n + m + 5 \) nodes)
Non-uniqueness of resolutions

<table>
<thead>
<tr>
<th>(Miranda)</th>
<th>(Szydlo)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$IV + I_0^*$: 1-2-4-2</td>
<td>$I_0^* + IV$: 1-2-3-2</td>
</tr>
</tbody>
</table>
Example of possible fibers of a fibration

(for a general cubic)
The case of SU(2)

Fiber structure

SU(2) \[ P_2 = 0 \]

I₂ \[ a_1 = 0 \]

I₃ \[ a_1 = 0 \]

III \[ P_2 = 0 \]

IV

codim 1
codim 2
codim 3
The case of SU(3)

Incidence geometry from the representation theory
The case of SU(3)

Tree of resolutions
The case of $SU(3)$

Fiber structure

$SU(3)$

$P_3 = 0$

$\text{codim 1}$

$a_1 = 0$

$\text{codim 2}$

$P_3 = 0$

$\text{codim 3}$

$a_1 = 0$
The case of SU(3)
The case of SU(3)

A perfect match!
The case of SU(4)

Incidence geometry
The case of SU(4)
The case of SU(4)

Incidence geometry
Tree of resolutions for SU(4)
The case of SU(4)
Tree of resolutions for SU(4)

Fiber structure

SU(4)

\[ P_4 = 0 \]

\[ a_1 = 0 \]

\[ a_1 = 0 \]

\[ P_4 = 0 \]

\[ a_{2,1}^2 - 4a_{4,2} = 0 \]

\[ a_{2} + 4a_{4,2} = 0 \]

\[ I_4 \]

\[ I_5 \]

\[ I_1^* \]

\[ I_0^* \]

\[ I_0^{*+} \]

codim 1  
codim 2  
codim 3
The case of SU(4)

A perfect match!
The case of SU(4)
The case of SU(4)
The case of SU(5)
THANK YOU!