Sphere Partition Functions, the Zamolodchikov Metric and Surface Operators

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with Le Floch, to appear
Introduction

• Recent years have seen dramatic progress in the **exact computation of partition functions** of supersymmetric field theories on **curved spaces**

• In geometries $S^1 \times M_d$, the partition function has a standard **Hilbert space** interpretation as a sum over states

$$Z[S^1 \times M_d] = \text{Tr}_\mathcal{H} \left[ (-1)^F e^{-\beta H} \right]$$

1) What does the partition function of a (S)CFT on $S^d$ compute?

• Physical Interpretation

• Ambiguities of $Z_{S^d}$

2) Sphere partition function $\Rightarrow$ M2$\subset$M5-brane surface operators
Sphere Partition Function in Conformal Manifold

- Exactly marginal operators $\int d^d x \lambda^i O_i$ define a family of CFTs spanning the conformal manifold $\mathcal{S}$:

  $$\lambda^i$$ are coordinates and $O_i$ are vectors fields in $\mathcal{S}$

- Conformal manifold $\mathcal{S}$ admits Riemannian metric: Zamolodchikov metric

  $$\langle O_i(x)O_j(0) \rangle_p = \frac{G_{ij}(p)}{x^{2d}} \quad p \in \mathcal{S}$$

- CFT can be canonically put on sphere for any $p \in \mathcal{S}$

- Sphere partition function is an infrared finite observable

- $Z_{S^d}$ is a probe of the conformal manifold $\mathcal{S}$
• Observable \( \langle \mathcal{O} \rangle_\lambda \) defined by expansion around reference CFT

\[
\langle \mathcal{O} \rangle_\lambda = \sum_k \frac{1}{k!} \langle \mathcal{O} \left( \int d^d x \sqrt{g} \lambda^i O_i(x) \right)^k \rangle
\]

• Integrated correlation functions have ultraviolet divergences

• Need to renormalize so that \( \langle \mathcal{O} \rangle_\lambda \) has a continuum limit

• The structure of divergences of sphere partition function is

\[
\log Z_{S^{2n}} = A_1[\lambda^i](r \Lambda_{UV})^{2n} + A_n[\lambda^i](r \Lambda_{UV})^2 + A[\lambda^i] \log(r \Lambda_{UV}) + F_{2n}[\lambda^i]
\]

\[
\log Z_{S^{2n+1}} = B_1[\lambda^i](r \Lambda_{UV})^{2n+1} + B_{n+1}[\lambda^i](r \Lambda_{UV}) + F_{2n+1}[\lambda^i]
\]

• Different renormalization schemes differ by diffeomorphism invariant local terms with \( \Delta \leq d \) constructed from background fields \( g_{mn}(x) \) and \( \lambda^i \to \lambda^i(x) \)

\[
\mathcal{L}(g_{mn}, \lambda^i)
\]
All power-law divergences can be tuned by appropriate counterterms.

In even dimensions:
- The finite piece $F_{2n}[\lambda^i]$ is ambiguous, there is a finite counterterm
  \[ \int d^{2n} x \sqrt{g} F_{2n}[\lambda^i] E_{2n} \]
- There is no local counterterm for the $A[\lambda^i] \log(r \Lambda_{UV})$ term
- Consistency requires that $A[\lambda^i] = A$, the A-type anomaly

In odd dimensions:
- There is no finite counterterm for $\text{Re}(F_{2n+1}[\lambda^i])$
- Consistency requires that $\text{Re}(F_{2n+1}[\lambda^i]) = \text{Re}(F_{2n+1})$
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Summary

• Unambiguous quantities $A$ and $\text{Re}(F_{2n+1})$ are constant along $S$

• $A$ and $\text{Re}(F_{2n+1})$ measure entanglement entropy across a sphere in the CFT

Casini, Huerta, Myers
SCFT Sphere Partition Functions

- Regulate the divergences in a **supersymmetric** way
- Preserve a “massive” subalgebra of superconformal algebra

\[ \{Q, Q\} = SO(d + 1) \oplus \text{R-symmetry} \]

This is the general supersymmetry algebra of a *massive* theory on $S^d$

- Counterterms are **diffeomorphism** and **supersymmetric** invariant
  \[ \implies \text{supergravity} \text{ counterterms} \]

- Realize $S^{2n}$ as supersymmetric background in a **supergravity** theory

  *Supergravity multiplet: $g_{mn}, \psi_m, \ldots$*

- Represent $\lambda^i$ as bottom component of a **superfield** $\Phi^i(x, \Theta)| = \lambda^i(x)$

- **Supergravity** invariant constructed from supergravity multiplet and $\Phi^i$

\[ \mathcal{L}(g_{mn}, \psi_m, \ldots; \lambda^i, \ldots) \]
Two Dimensional $\mathcal{N} = (2, 2)$ SCFTs

- Includes worldsheet description of string theory on Calabi-Yau manifolds
- Conformal manifold $\mathcal{S}$ is Kähler and locally $\mathcal{S}_c \times \mathcal{S}_{tc}$
- $\nexists$ an $\mathcal{N} = (2, 2)$ superconformal invariant regulator. $\exists$ two massive $\mathcal{N} = (2, 2)$ subalgebras on $S^2$

\[
SU(2|1)_A \xleftrightarrow{\text{mirror}} SU(2|1)_B
\]

- Defines partition functions $Z_A$ and $Z_B$

- Compute the exact Kähler potential $K$ on the conformal manifold

$$Z_A = e^{-K_{tc}} \quad Z_B = e^{-K_c}$$

- Partition function subject to ambiguity under Kähler transformations

$$K \rightarrow K + \mathcal{F}(\lambda^i) + \overline{\mathcal{F}}(\overline{\lambda}^i)$$

$\mathcal{F}$ is a holomorphic function instead of an arbitrary real function of the moduli
• Kähler ambiguity counterterm in Type A/B 2d $\mathcal{N} = (2, 2)$ supergravity. Supergravities gauge either $U(1)_V$ or $U(1)_A$ R-symmetry.

• Coordinates in $S_c$ are bottom components of chiral multiplets $\Phi^i$.

• Coordinates in $S_{tc}$ are bottom components of twisted chiral multiplets $\Omega^i$.

• The $SU(2|1)_B$ Kähler ambiguity is due to the supergravity coupling

$$\int d^2x d^2\Theta \varepsilon R \mathcal{F} (\Phi^i) + c.c \supset \frac{1}{r^2} \int d^2 x \sqrt{g} \mathcal{F} (\lambda^i) + c.c$$

$\mathcal{F}$: holomorphic function

$R$: chiral superfield containing $\mathcal{R}$ as top component

$\varepsilon$: chiral density superspace measure

• The $SU(2|1)_A$ Kähler ambiguity is parametrized by

$$\int d^2x d\Theta^+ d\tilde{\Theta}^- \hat{\varepsilon} F \mathcal{F} (\Omega^i) + c.c$$
4d $\mathcal{N} = 2$ SCFTs

- Conformal Manifold $\mathcal{S}$ of 4d $\mathcal{N} = 2$ SCFTs is Kähler
- SCFT on $S^4$ can be deformed by exactly marginal operators

$$\int d^4 x \sqrt{g} \sum_i (\tau_i C_i + \bar{\tau}_i \bar{C}_i)$$

$C_i$ : top component of 4d $\mathcal{N} = 2$ chiral multiplet with bottom component $A_i$

$\tau_i$ : coordinates on conformal manifold $\mathcal{S}$
- Regulate divergences of $Z_{S^4}$ in an $OSp(2|4) \subset SU(2,2|2)$ invariant way
- Calculate by supersymmetric localization or using Ward identity

$$\partial_i \partial_{\bar{j}} \log Z_{S^4} = \left\langle \int_{S^4} d^4 x \sqrt{g} C_i(x) \int_{S^4} d^4 y \sqrt{g} \bar{C}_{\bar{j}}(y) \right\rangle = \left\langle A_i(N) \bar{A}_{\bar{j}}(S) \right\rangle = G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$$

- $Z_{S^4}$ of 4d $\mathcal{N} = 2$ SCFTs computes the Kähler potential on $\mathcal{S}$

$$Z_{S^4} = e^{K/12}$$
• How about 4d $\mathcal{N} = 1$ SCFTs?

• Conformal Manifold $\mathcal{S}$ is Kähler

• Partition Function regulated in an $OSp(1|4) \subset SU(2,2|1)$ invariant way

• $\exists$ 4d $\mathcal{N} = 1$ (old minimal) supergravity finite counterterm

$$\int d^4 x \int d^2 \Theta \varepsilon (\bar{D}^2 - 8R)R \bar{R} F(\Phi^i, \bar{\Phi}^\bar{i}) \supset \frac{1}{r^4} \int d^4 x \sqrt{g}F(\lambda^i, \bar{\lambda}^\bar{i})$$

$F$ arbitrary $\implies Z_{S^4}$ for $\mathcal{N} = 1$ SCFTs is ambiguous
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Summary

• $S^{2n}$ partition function of SCFTs may have reduced space of ambiguities

• Sphere partition functions of $2d \mathcal{N} = (2, 2)$ and $4d \mathcal{N} = 2$ SCFTs capture the exact Kähler potential on their conformal manifold
Surface Operators and M2-branes

- M2-branes ending on $N_f$ M5-branes

\[
\begin{array}{c|cccccc}
\text{M5} & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
\text{M2} & 0 & 1 &  &  &  & 6
\end{array}
\]

insert a surface operator in the 6d $\mathcal{N} = (2, 0) \ A_{N_f-1}$ SCFT

- Surface operators labeled by a representation $\mathcal{R}$ of $SU(N_f)$

- M5-branes wrapping a punctured Riemann surface $C$ realize a large class of 4d $\mathcal{N} = 2$ theories (class S) Gaiotto

- M2-branes ending on $N_f$ M5-branes insert a surface operator in the corresponding 4d $\mathcal{N} = 2$ theory

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\begin{array}{c|cccccc}
\text{M5} & 0 & 1 & 2 & 3 & 4 & 5 \\
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• **Surface operators** in 4d gauge theories

  • Order parameters that go beyond the Wilson-’t Hooft criteria
  
  • Can be described by coupling 2d defect dof to the bulk gauge theory
  
  • Coupled 4d/2d system can exhibit new dynamics and dualities

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- M2-brane surface operators preserve a 2d $\mathcal{N} = (2, 2)$ subalgebra of 4d $\mathcal{N} = 2$

- We have identified the 2d gauge theories corresponding to M2-branes

\[ \begin{align*}
N_1 - N_2 &
N_{n-1} - N_n \\
N_2 - N_3 &
\end{align*} \]
Surface operator obtained by identifying the $SU(N_f) \times SU(N_f) \times U(1)$ symmetry of the 2d gauge theory with a corresponding gauge or global symmetry of 4d $\mathcal{N} = 2$ theory.

A superpotential on the defect couples 2d fields to 4d fields.
- $S^4_b$ partition function of $\mathcal{T}_C$ is captured by Toda CFT correlator in $C$ Pestun AGT

\[ Z[\mathcal{T}_C] \leftrightarrow \]

- Conjecturally, a degenerate puncture describes a surface operator AGGTV
• $S^4_b$ partition function of $\mathcal{T}_C$ is captured by Toda CFT correlator in $C$

\[ Z[\mathcal{T}_C] \leftrightarrow \]

• Conjecturally, a degenerate puncture describes a surface operator

\[ Z_{S^2 \subset S^4_b}^{\mathcal{R}[\Omega]} \leftrightarrow \]

\[ S^4_b \text{ partition function of } \mathcal{T}_C \quad = \quad \text{Toda CFT correlator on } C \]

+ our 2d gauge theory on $S^2$

labelled by $\mathcal{R}(\Omega)$

= + extra degenerate

with momentum $\alpha = -b\Omega$
- $S^4_b$ partition function of $\mathcal{T}_C$ is captured by Toda CFT correlator in $C$
- Conjecturally, a degenerate puncture describes a surface operator $\text{AGGTV}$
- $S^4_b$ partition function of $\mathcal{T}_C$ + our 2d gauge theory on $S^2$ labelled by $\mathcal{R}(\Omega)$ = Toda CFT correlator on $C$ + extra degenerate with momentum $\alpha = -b\Omega$
- We explicitly verified this for the 4d $\mathcal{N} = 2$ theory associated to the trinion by using exact formulae for the $S^2$ partition function of 2d $\mathcal{N} = (2, 2)$ theories $\text{Benini,Cremonesi; Doroud,J.G,Le Floch,Lee}$
Gauge Theory Dualities as Toda CFT Symmetries

- Through our identification between 2d gauge theories and Toda CFT

  \[ \text{Toda CFT Symmetries} \implies 4d/2d \text{ and } 2d \text{ Gauge Theory Dualities} \]

\[ \begin{array}{c}
\text{2d Seiberg and Kutasov–Schwimmer dualities}
\end{array} \]

\[ \begin{array}{c}
\text{2d Seiberg and } (2, 2)^* \text{ dualities for quivers}
\end{array} \]
<table>
<thead>
<tr>
<th>Duality</th>
<th>Quiver</th>
<th>$W$</th>
<th>Dual parameters</th>
</tr>
</thead>
</table>
| Seiberg             | ![Seiberg Quiver](image) | 0                    | $N^D = N_f - N$  
$z^D = z$, $m^D = i/2 - m$ |
| $(2, 2)^*$-like     | ![$(2, 2)^*$-like Quiver](image) | $\sum_t \tilde{q}_t X^{l_t} q_t$ | $N^D = \sum_t l_t - N$  
$z^D = z^{-1}$, $m^D = m$ |
| Kutasov–Schwimmer   | ![Kutasov–Schwimmer Quiver](image) | $\text{Tr} X^{l+1}$ | $N^D = lN_f - N$  
$z^D = z$, $m^D = i/2 - m$ |
Conclusion

• In nonsupersymmetric CFTs, $F_{2n+1}$ and A-anomaly are the scheme independent pieces of sphere partition functions

• Sphere partition functions of 2d $\mathcal{N} = (2, 2)$ and 4d $\mathcal{N} = 2$ SCFTs capture the exact Kähler potential on their conformal manifold

\[ Z_A = e^{-K_{tc}} \quad Z_B = e^{-K_c} \quad Z_{S^4} = e^{K/12} \]

• Identified supergravity realization of Kähler transformation ambiguities

• Gave microscopic description of all M2-brane surface operators

• Dualities of 2d $\mathcal{N} = (2, 2)$ theories realized in Toda CFT