The Exact Renormalization Group and Higher Spin Holography

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Introduction

- An appealing aspect of holography is its interpretation in terms of the renormalization group of quantum field theories — the ‘radial coordinate’ is a geometrization of the renormalization scale — Hamilton-Jacobi theory of the radial quantization is expected to play a central role.

  e.g., [de Boer, Verlinde '99, Skenderis '02, Heemskerk & Polchinski '10, Faulkner, Liu & Rangamani '10 ...]

- usually this is studied from the bulk side, as the QFT is typically strongly coupled

- here, we will approach the problem directly from the field theory side, using the Wilson-Polchinski exact renormalization group around (initially free) field theories [Douglas, Mazzucato & Razamat '10]

- of course, we can’t possibly expect to find a purely gravitational dual

  ▶ but there is some hope given the conjectured dualities between higher spin theories and vector models (for example).

  [Klebanov & Polyakov '02, Sezgin & Sundell '02, Leigh & Petkou '03] [Vasiliev '96, '99, '12] [de Mello Koch et al...]

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The Exact Renormalization Group (ERG)

- Polchinski ’84: formulated field theory path integral by introducing a regulator given by a cutoff function accompanying the fixed point action (i.e., the kinetic term).

$$Z = \int [d\phi] e^{-\int \phi K_F^{-1}(-\Box/M^2)\Box \phi - S_{\text{int}}[\phi]}$$

- this equation describes how the couplings must depend on the RG scale in order that the partition function be independent of the cutoff.

- can apply similar methods to correlation functions, and thus obtain exact Callan-Symanzik equations as well.
in this form, the ERG equations will be inconvenient — instead of moving the cutoff, we would like to fix the cutoff and move a renormalization scale ($z$)

the ERG equations are first order equations, while bulk EOM are often second order

solutions of such equations though are interpreted in terms of sources and vevs — the expected H-J structure implies that these should be thought of as canonically conjugate in radial quantization

thus, we anticipate that the ERG equations for sources and vevs should be thought of as first-order Hamilton equations in the bulk
Locality is Over-Rated

- higher spin theories possess a huge gauge symmetry
- if the theory is really holographic, we expect to be able to identify this symmetry within the dual field theory
- unbroken higher spin symmetry implies an infinite number of conserved currents — one can hardly expect to find a local theory
- indeed, free field theories have a huge non-local symmetry
- e.g., $N$ Majoranas in $2 + 1$

$$S_0 = \int_{x,y} \tilde{\psi}^m(x) \gamma^\mu P_{F;\mu}(x, y) \psi^m(y) \equiv \int \tilde{\psi}^m \cdot \gamma^\mu P_{F;\mu} \cdot \psi^m$$

$$P_{F;\mu}(x, y) = K_F^{-1}(-\Box/M^2)\partial^{(x)}_\mu \delta(x - y)$$

- we also include sources for ‘single-trace’ operators

$$S_{int} = U + \frac{1}{2} \int_{x,y} \tilde{\psi}^m(x) \left(A(x, y) + \gamma^\mu W_\mu(x, y)\right) \psi^m(y)$$
The $O(L_2(\mathbb{R}^d))$ Symmetry

- Bi-local sources collect together infinite sets of local operators, obtained by expanding near $x \to y$

$$A(x, y) = \sum_{s=0}^{\infty} A^{a_1 \ldots a_s}(x) \partial_{a_1}^{(x)} \cdots \partial_{a_s}^{(x)} \delta(x - y)$$

- Now we consider the following bi-local map of elementary fields

$$\psi^m(x) \mapsto \int_y \mathcal{L}(x, y) \psi^m(y) = \mathcal{L} \cdot \psi^m(x)$$

- We look at the action

$$S \rightarrow \bar{\psi}^m \cdot \mathcal{L}^T \cdot \left[ \gamma^\mu (P_{F;\mu} + W_\mu) + A \right] \cdot \mathcal{L} \cdot \psi^m$$

$$= \bar{\psi}^m \cdot \gamma^\mu \mathcal{L}^T \cdot \mathcal{L} \cdot P_{F;\mu} \cdot \psi^m$$

$$+ \bar{\psi}^m \cdot \left[ \gamma^\mu (\mathcal{L}^T \cdot [P_{F;\mu}, \mathcal{L}] + \mathcal{L}^T \cdot W_\mu \cdot \mathcal{L}) + \mathcal{L}^T \cdot A \cdot \mathcal{L} \right] \cdot \psi^m$$
The $O(L_2)$ Symmetry

Thus, if we take $\mathcal{L}$ to be orthogonal,

$$\mathcal{L}^T \cdot \mathcal{L}(x, y) = \int_z \mathcal{L}(z, x)\mathcal{L}(z, y) = \delta(x, y),$$

the kinetic term is invariant, while the sources transform as

**$O(L_2)$ gauge symmetry**

$$W_\mu \mapsto \mathcal{L}^{-1} \cdot W_\mu \cdot \mathcal{L} + \mathcal{L}^{-1} \cdot [P_{F;\mu}, \mathcal{L}]$$

$$A \mapsto \mathcal{L}^{-1} \cdot A \cdot \mathcal{L}$$

We interpret this to mean that the source $W_\mu(x, y)$ is the $O(L_2)$ connection, with the regulated derivative $P_{F;\mu}$ playing the role of derivative.
The $O(L_2)$ Ward Identity

- But this was a trivial operation from the path integral point of view, and so we conclude that there is an exact Ward identity

$$Z[M, g(0), W_\mu, A] = Z[M, g(0), \mathcal{L}^{-1} \cdot W_\mu \cdot \mathcal{L} + \mathcal{L}^{-1} \cdot P_{F;\mu} \cdot \mathcal{L}, \mathcal{L}^{-1} \cdot A \cdot \mathcal{L}]$$

- this is the usual notion of a background symmetry: a transformation of the elementary fields is compensated by a change in background

- more generally, we can turn on sources for arbitrary multi-local multi-trace operators — the sources will generally transform tensorially under $O(L_2)$
The $O(L_2)$ Symmetry

- **Punch line:** the $O(L_2)$ transformation leaves the (regulated) fixed point action invariant. $D_\mu = P_{F;\mu} + W_\mu$ plays the role of covariant derivative.

- More precisely, the free fixed point corresponds to any configuration

\[(A, W_\mu) = (0, W^{(0)}_\mu)\]

where $W^{(0)}$ is any flat connection, $dW^{(0)} + W^{(0)} \wedge W^{(0)} = 0$

- It is therefore useful to split the full connection as

\[W_\mu = W^{(0)}_\mu + \hat{W}_\mu\]

- will choose it to be invariant under the conformal algebra
  - $W^{(0)}$ is a flat connection associated with the fixed point
  - $A, \hat{W}$ are operator sources, transforming tensorially under $O(L_2)$
The \( CO(L_2) \) symmetry

- We generalize \( O(L_2) \) to include scale transformations

\[
\int_z \mathcal{L}(z, x)\mathcal{L}(z, y) = \lambda^{2\Delta}\delta(x - y)
\]

- This is a symmetry (in the previous sense) provided we also transform the metric, the cutoff and the sources

\[
g(0) \mapsto \lambda^2 g(0), \quad M \mapsto \lambda^{-1} M
\]

\[
A \mapsto \mathcal{L}^{-1} \cdot A \cdot \mathcal{L}
\]

\[
W_\mu \mapsto \mathcal{L}^{-1} \cdot W_\mu \cdot \mathcal{L} + \mathcal{L}^{-1} \cdot [P_{F;\mu}, \mathcal{L}].
\]

- A convenient way to keep track of the scale is to introduce the conformal factor \( g(0) = \frac{1}{z^2} \eta \). Then \( z \mapsto \lambda^{-1}z \). This \( z \) should be thought of as the renormalization scale.
The Renormalization group

- To study RG systematically, we proceed in two steps:

**Step 1**: Lower the cutoff $M \mapsto \lambda M$, by integrating out the “fast modes”

\[
Z[M, z, A, W] = Z[\lambda M, z, \tilde{A}, \tilde{W}] \quad \text{(Polchinski)}
\]

**Step 2**: Perform a $CO(L_2)$ transformation to bring the cutoff back to $M$, but in the process changing $z \mapsto \lambda^{-1} z$

\[
Z[\lambda M, z, \tilde{A}, \tilde{W}] = Z[M, \lambda^{-1} z, \mathcal{L}^{-1} \cdot \tilde{A} \cdot \mathcal{L}, \mathcal{L}^{-1} \cdot \tilde{W} \cdot \mathcal{L} + \mathcal{L}^{-1} \cdot [P_F, \mathcal{L}]]
\]

- We can now compare the sources at the same cutoff, but different $z$. Thus, $z$ becomes the natural flow parameter, and we can think of the sources as being $z$-dependent. (Thus we have the Wilson-Polchinski formalism extended to include both a cutoff and an RG scale — required for a holographic interpretation).
Infinitesimal version: RG equations

- Infinitesimally, we parametrize the $CO(L_2)$ transformation as
  \[ \mathcal{L} = 1 + \varepsilon z W_z \]
- $W_z$ should be thought of as the $z$-component of the connection.
- The RG equations become
  \[ A(z + \varepsilon z) = A(z) + \varepsilon z [W_z, A] + \varepsilon z \beta^{(A)} + O(\varepsilon^2) \]
  \[ W_\mu(z + \varepsilon z) = W_\mu(z) + \varepsilon z [P_{F;\mu} + W_\mu, W_z] + \varepsilon z \beta^{(W)}_\mu + O(\varepsilon^2) \]
- The beta functions are *tensorial*, and quadratic in $A$ and $\hat{W}$.
- Thus, RG extends the sources $A$ and $W$ to bulk fields $\mathcal{A}$ and $\mathcal{W}$. 
RG equations

- Comparing terms linear in $\varepsilon$ gives

$$\partial_z W^{(0)}_\mu - [P_{F;\mu}, W^{(0)}_z] + [W^{(0)}_z, W^{(0)}_\mu] = 0$$

$$\partial_z A + [W_z, A] = \beta^{(A)}$$

$$\partial_z W_\mu - [P_{F;\mu}, W_z] + [W_z, W_\mu] = \beta^{(W)}_\mu$$

- These equations are naturally thought of as being part of fully covariant equations (e.g., the first is the $z_\mu$ component of a bulk 2-form equation, where $d \equiv dx^\mu P_{F;\mu} + dz\partial_z$.)

$$d W^{(0)} + W^{(0)} \wedge W^{(0)} = 0$$

$$d A + [W, A] = \beta^{(A)}$$

$$d W + W \wedge W = \beta^{(W)}$$

$$D\beta^{(A)} = \left[\beta^{(W)}, A\right], \quad D\beta^{(W)} = 0$$

- The resulting equations are then diff invariant in the bulk.
Similarly, one can extract exact Callan-Symanzik equations for the $z$-dependence of $\Pi(x, y) = \langle \tilde{\psi}(x)\psi(y) \rangle$, $\Pi^\mu(x, y) = \langle \tilde{\psi}(x)\gamma^\mu\psi(y) \rangle$. These extend to bulk fields $\mathcal{P}, \mathcal{P}^A$.

The full set of equations then give rise to a phase space formulation of a dynamical system — $(A, \mathcal{P})$ and $(\mathcal{W}_A, \mathcal{P}^A)$ are canonically conjugate pairs from the point of view of the bulk.

If we identify $Z = e^{iS_{HJ}}$, then a fundamental relation in H-J theory is

$$\frac{\partial}{\partial Z} S_{HJ} = -\mathcal{H}$$

We can thus read off this Hamiltonian — it can be thought of as the output of the ERG analysis.

there is a corresponding action $S_{HJ}$ for this higher spin theory, written in terms of phase space variables.
Hamilton-Jacobi Structure

- We interpret this phase space theory as the higher spin gauge theory
- this theory is written as a gauge theory on a spacetime, topology \( \sim \mathbb{R}^d \times \mathbb{R}^+ \)
- we’ve identified a specific flat connection \( \mathcal{W}^{(0)} \) representing the free fixed point

\[
\mathcal{W}^{(0)}(x, y) = -\frac{dz}{z} D(x, y) + \frac{dx^\mu}{z} P_\mu(x, y)
\]

where \( P_\mu(x, y) = \partial_\mu^{(x)} \delta(x - y) \) and \( D(x, y) = (x^\mu \partial_\mu^{(x)} + \Delta) \delta(x - y) \).

- This connection is equivalent to the vielbein and spin connection of \( \text{AdS}_{d+1} \).
  - \( \mathcal{W}^{(0)} \) is invariant under the conformal algebra \( o(2, d) \subset co(L_2) \).
Geometry: The Infinite Jet bundle

- We can put the non-local transformation \( \psi(x) \mapsto \int_y L(x, y) \psi(y) \) in more familiar terms by introducing the notion of a jet bundle.

- The simple idea is that we can think of a differential operator \( L(x, y) \) as a matrix by "prolongating" the field

\[
\psi^m(x) \mapsto \left( \psi^m(x), \frac{\partial \psi^m}{\partial x_\mu}(x), \frac{\partial^2 \psi^m}{\partial x_\mu \partial x_\nu}(x) \ldots \right) \quad "\text{jet}" \]

- Then, differential operators, such as \( P_\mu(x, y) = \partial_\mu^{(x)} \delta(x - y) \) are interpreted as matrices \( P_\mu \) that act on these vectors.

- The bi-local transformations can be thought of as local gauge transformations of the jet bundle.

- The gauge field \( W \) is a connection 1-form on the jet bundle, while \( A \) is a section of its endomorphism bundle.
Other Examples

- the 2 + 1 Majorana model is presumably equivalent to the Vasiliev B-model
- extensions to higher dimensions require additional sources for $\tilde{\psi} \gamma^{ab} \psi$, ....
- $N$ complex bosons: construct in similar terms
  \[ S = \int \tilde{\phi}_m \cdot \left( \left[ D_{F;\mu} + W_\mu \right]^2 + B \right) \cdot \phi^m \]
- The ERG equations give rise to an ‘A-model’ in any dimension.
  \[ [RGL, O. Parrikar, A.B. Weiss, to appear.]
- Here though there is an extra background symmetry
  \[ Z[M, z, B, W_\mu^{(0)}, \widehat{W}_\mu + \Lambda_\mu] = Z[M, z, B + \{\Lambda^\mu, D_\mu\} + \Lambda_\mu \cdot \Lambda^\mu, W_\mu^{(0)}, \widehat{W}_\mu] \]
- this background symmetry allows for fixing $W_\mu \rightarrow W_\mu^{(0)}$, and the corresponding transformed $B$ sources all single-trace currents.
The Bulk Action and Correlation Functions

- For the bosonic theory, the bulk phase space action is

\[ I = \int dz \, \text{Tr} \left\{ \mathcal{P}^I \cdot \left( \mathcal{D}_I \mathcal{B} - \beta^{(0)}_I \right) + \mathcal{P}^{IJ} \cdot \mathcal{F}^{(0)}_{IJ} + N \, \Delta_B \cdot \mathcal{B} \right\} \]

- Here \( \Delta_B \) is a derivative with respect to \( M \) of the cutoff function.

- As in any holographic theory, we solve the bulk equations of motion in terms of boundary data, and obtain the on-shell action, which encodes the correlation functions of the field theory.

- It is straightforward to carry this out exactly for the free fixed point.

- Here we have

\[ I_{o.s.} = N \int \Delta_B \cdot \mathcal{B}_{o.s.} \]

where now \( \mathcal{B}_{o.s.} \) is the bulk solution.
The Bulk Action and Correlation Functions

- The RG equation

\[
\left[ D_z^{(0)}, \mathcal{B} \right] = \beta_z^{(B)} = \mathcal{B} \cdot \Delta_B \cdot \mathcal{B}
\]

can be solved iteratively

\[
\mathcal{B} = \alpha \mathcal{B}_1 + \alpha^2 \mathcal{B}_2 + \ldots,
\]

\[
\begin{align*}
\left[ D_z^{(0)}, \mathcal{B}_1 \right] &= 0 \\
\left[ D_z^{(0)}, \mathcal{B}_2 \right] &= \mathcal{B}_1 \cdot \Delta_B \cdot \mathcal{B}_1 \\
\left[ D_z^{(0)}, \mathcal{B}_3 \right] &= \mathcal{B}_2 \cdot \Delta_B \cdot \mathcal{B}_1 + \mathcal{B}_1 \cdot \Delta_B \cdot \mathcal{B}_2 \\
&\vdots
\end{align*}
\]
The Bulk Action and Correlation Functions

- The first equation (1) is homogeneous and has the solution

\[ \mathcal{B}_1(z; x, y) = \int_{x', y'} K^{-1}(z; x, x') b_0(x', y') K(z; y', y) \]

where we have defined the boundary-to-bulk Wilson line

\[ K(z) = P. \exp \int_\epsilon^z dz' \, \mathcal{W}_z^{(0)}(z') \]

with the boundary being placed at \( z = \epsilon \).

- \( b_0 \) has the interpretation of a boundary source
- this can then be inserted into the second order equation and the whole system solved iteratively
At \( k^{th} \) order, one finds a contribution to the on-shell action

\[
I_{o.s.}^{(k)} = N \int_{\epsilon}^{\infty} dz_1 \int_{\epsilon}^{z_1} dz_2 \cdots \int_{\epsilon}^{z_{k-1}} dz_k \\
\times \text{Tr} \ H(z_1) \cdot b(0) \cdot H(z_2) \cdot b(0) \cdot \cdots \cdot H(z_k) \cdot b(0) \\
+ \text{permutations}
\]

where \( H(z) \equiv K^{-1}(z) \cdot \Delta_B(z) \cdot K(z) = \partial_z g(z) \)

The Witten diagram for the bulk on-shell action at third order.
The Bulk Action and Correlation Functions

- The $z$-integrals can be performed trivially, resulting in

$$I_{o.s.}^{(k)} = \frac{N}{k} \text{Tr} \left( g_{(0)} \cdot b_{(0)} \right)^k$$

where $g_{(0)} = g(\infty)$ is the boundary free scalar propagator.

- These can be resummed, resulting in

$$Z[b_{(0)}] = \det^{-N} \left( 1 - g_{(0)} b_{(0)} \right)$$

which is the exact generating functional for the free fixed point.

- Thus, this holographic theory does everything that it can for us.
really, this analysis should be thought of within a larger system, in which field theory interactions are turned on.

For example, if we turn on all multi-local multi-trace interactions, we obtain an infinite set of ERG equations – the bulk theory now contains an infinite number of conjugate pairs.

The Gaussian theory is a consistent truncation of this more general theory, in which the higher spin gauge symmetry remains unbroken.

We expect that there are other solutions of the full ERG equations with other boundary data specified (such as a 4-point coupling), corresponding to other fixed points.

An example is the W-F large $N$ critical point.