Unity of tree-level superstring amplitudes

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(a) *Structure of string amplitudes has deep impact on the form and organization of quantum field theory amplitudes.*

\[
M_{FT}(1, 2, 3, 4) = s_{12} A_{FT}(1, 2, 3, 4) \tilde{A}_{FT}(1, 2, 4, 3)
\]

graviton amplitudes = (gauge amplitudes) × (gauge amplitudes)

many relations in field-theory emerge from properties of string world-sheet:

monodromy on world-sheet yield KLT, BCJ, ... relations
(b) (Quite recent) mathematical concepts from/in string amplitudes elements of arithmetic algebraic geometry (e.g. motives, symbols, coproduct, …) underlay the structure of superstring amplitudes

\[ \alpha' - \text{expansion} \]
open, closed superstring amplitude

decomposition of motivic MZVs
Drinfeld, Deligne associators
Amplitudes are key players in establishing string dualities. Unexpected relations between open and closed string amplitudes (beyond KLT) may emerge (to all orders in $\alpha'$, i.e., beyond BPS).

Based on:


Heterotic gauge amplitudes as single-valued type I gauge amplitudes

Tree-level N-point type I open superstring gauge amplitude:

\[ A^I_N = (g^I_{YM})^{N-2} \sum_{\Pi \in S_N/\mathbb{Z}_2} \text{Tr}(T^{a_{\Pi(1)}} \cdots T^{a_{\Pi(N)}}) A^I(\Pi(1), \ldots, \Pi(N)) \]

Tree-level N-point heterotic closed string gauge amplitude:

\[ A^\text{HET}_N = (g^\text{HET}_{YM})^{N-2} \sum_{\Pi \in S_N/\mathbb{Z}_2} \text{Tr}(T^{a_{\Pi(1)}} \cdots T^{a_{\Pi(N)}}) A^\text{HET}(\Pi(1), \ldots, \Pi(N)) + O(1/N_c^2) \]

Result:

\[ A^\text{HET}(\Pi) = \text{sv} \left( A^I(\Pi) \right) \]

sv = single-valued projection
This will be generalized to any closed string amplitude: 
**closed string amplitudes as single-valued open string amplitudes**

at the level of world-sheet integrals:

\[ \int_{C} d^{2}z \frac{|z|^{2s} |1 - z|^{2u}}{z (1 - z) \bar{z}} = \text{sv} \left( \int_{0}^{1} dx \ x^{s-1} (1 - x)^{u} \right) \]

\[ \frac{1}{s} \frac{\Gamma(s) \Gamma(u) \Gamma(t)}{\Gamma(-s) \Gamma(-u) \Gamma(-t)} = \text{sv} \left( \frac{\Gamma(s) \Gamma(1 + u)}{\Gamma(1 + s + u)} \right) \]

No KLT relations necessary!

KLT:

\[ \int_{C} d^{2}z \frac{|z|^{2s} |1 - z|^{2u}}{z (1 - z) \bar{z}} = \sin(\pi u) \left( \int_{0}^{1} x^{s-1} (1 - x)^{u-1} \right) \left( \int_{1}^{\infty} x^{t-1} (1 - x)^{u} \right) \]
Type I gauge amplitude:

\[ \mathcal{A}^I(\pi) = (-1)^{N-3} \sum_{\sigma \in S_{N-3}} \sum_{\rho \in S_{N-3}} Z_\pi(\rho) \ S[\rho|\sigma] \ A_{YM}(\sigma) \]

Mafra, Schlotterer, St.St. (2011)
Broedel, Schlotterer, St.St. (2013)

fundamental world-sheet disk integrals:
iterated real integral on \( \mathbb{RP}^1 \setminus \{0, 1, \infty\} \)

Heterotic gauge amplitude:

\[ \mathcal{A}^{\text{HET}}(\rho) = (-1)^{N-3} \sum_{\sigma \in S_{N-3}} \sum_{\bar{\rho} \in S_{N-3}} J[\rho | \bar{\rho}] \ S[\bar{\rho}|\sigma] \ A_{YM}(\sigma) \]

Taylor, St.St. (2014)

\[ J = s_v(Z) \]

fundamental world-sheet sphere integrals:
integral on \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \)

Supergravity N-graviton amplitude:

\[ \mathcal{M}_{FT}(1, \ldots, N) = (-1)^{N-3} \kappa^{N-2} \sum_{\sigma \in S_{N-3}} \sum_{\rho \in S_{N-3}} \tilde{A}_{YM}(\rho) \ S[\rho|\sigma] \ A_{YM}(\sigma) \]

\[ S = \text{KLT kernel} \]
\[ S = \text{KLT kernel} \quad S[\rho|\sigma] := S[\rho(2, \ldots, N-2)|\sigma(2, \ldots, N-2)] \]

\[
= \prod_{j=2}^{N-2} \left( s_{1,j\rho} + \sum_{k=2}^{j-1} \theta(j\rho, k\rho) s_{j\rho,k\rho} \right) \\
\] 

\[ s_{ij} = \alpha'(k_i + k_j)^2 \]

Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove (2010)

\[ \text{side note: in super-twistor or dual super-twistor space:} \]

\[ \tilde{A}_{YM}(\rho) \simeq Z_\pi(\rho) \]

subject to Mellin transformation

gives rise to superstring/supergravity  
Mellin correspondence

Taylor, St.St. (2013)
Multiple zeta-values in superstring theory

Disk integrals: iterated real integral on $\mathbb{RP}^1\setminus\{0, 1, \infty\}$

Expand w.r.t. $\alpha'$:

$$Z_{23}(23) = V_{\text{CKG}}^{-1} \int_{z_i < z_{i+1}} \left( \prod_{j=1}^{5} dz_j \right) \prod_{1 \leq i < j \leq 5} |z_{ij}|^{s_{ij}}$$

$$= \frac{1}{s_{12}s_{45}} + \frac{1}{s_{23}s_{45}} + \zeta(2) \left( 1 - \frac{s_{34}}{s_{12}} - \frac{s_{12}}{s_{45}} - \frac{s_{23}}{s_{45}} - \frac{s_{51}}{s_{23}} \right) + \mathcal{O}(\alpha')$$

**Terasoma & Brown:** the coefficients of the Taylor expansion of the Selberg integrals w.r.t. the variables $s_{ij}$ can be expressed as linear combinations of MZVs over $\mathbb{Q}$

$$\zeta_{n_1, \ldots, n_r} := \zeta(n_1, \ldots, n_r) = \sum_{0 < k_1 < \ldots < k_r} \prod_{l=1}^{r} k_l^{-n_l}, \quad n_l \in \mathbb{N}^+, \quad n_r \geq 2,$$

Commutative graded $\mathbb{Q}$-algebra:

$$\mathcal{Z} = \bigoplus_{k \geq 0} \mathcal{Z}_k, \quad \dim_{\mathbb{Q}}(\mathcal{Z}_N) = d_N$$

with: $d_N = d_{N-2} + d_{N-3}$, $d_0 = 1$, $d_1 = 0$, $d_2 = 1$, $\ldots$ (Zagier)
Recall: $\mathcal{A}^I(\pi) = (-1)^{N-3} Z S A_{YM}(\sigma)$ \hspace{1cm} $\pi, \sigma \in S_{N-3}$

Define: $F_{\pi\sigma} := (-1)^{N-3} \sum_{\rho \in S_{N-3}} Z_{\pi}(\rho) S[\rho|\sigma]$ \hspace{1cm} $F = \text{period matrix}$

$$F(\alpha') = P \, Q \, \exp \left\{ \sum_{n \geq 1} \zeta_{2n+1} M_{2n+1} \right\}$$

$$Q = 1 + \frac{1}{5} \zeta_{3,5} [M_5, M_3] + \left\{ \frac{3}{14} \zeta_5^2 + \frac{1}{14} \zeta_{3,7} \right\} [M_7, M_3] + \left\{ 9 \zeta_2 \zeta_9 + \frac{6}{25} \zeta_2^2 \zeta_7 - \frac{4}{35} \zeta_2^3 \zeta_5 + \frac{1}{5} \zeta_{3,3,5} \right\} [M_3, [M_5, M_3]] + \ldots$$

This form exactly appears in F. Brown’s decomposition of motivic multiple zeta values!

Schlotterer, Stieberger, arXiv:1205.1516
To explicitly describe the structure of the algebra $\mathcal{Z}$, MZVs are replaced by their symbols (or motivic MZVs) $\zeta^m \in \mathcal{H}$.

Goncharov, Brown:

$$\zeta^m \longrightarrow \phi(\zeta^m)$$

normalized by:

$$\phi(\zeta^m_n) = f_n, \quad n \geq 2$$

$$\mathcal{H} \longrightarrow \mathcal{U}$$

$$\mathcal{U} = \mathbb{Q}\langle f_3, f_5, \ldots \rangle \otimes_{\mathbb{Q}} \mathbb{Q}[f_2]$$

Map $\phi$ sends every motivic MZV $\xi \in \mathcal{H}$ to a non-commutative polynomial in $f_i$'s.

E.g.:

$$\phi \left( \zeta_3^m \zeta_5^m \right) = f_3 \sqcup f_5 \equiv f_3 f_5 + f_5 f_3$$

$$\phi \left( \zeta_{3,5}^m \right) = -5 f_5 f_3$$

F. Brown (2012)

Motivic open superstring amplitude:

$$\phi(A^m) = \left( \sum_{k=0}^{\infty} f_2^k \ P_{2k} \right) \left( 1 - \sum_{k=1}^{\infty} f_{2k+1} \ M_{2k+1} \right)^{-1} A_{YM}$$

Schlotterer, Stieberger, arXiv:1205.1516
There is a natural homomorphism:

F. Brown (2013): \( \text{sv} : w \mapsto \text{sv}(w) := \sum_{uv=w} u \sqcup \tilde{v} \)

\[
\text{sv}(f_a) = 2 f_a \\
\text{sv}(f_a f_b) = 2 f_a \sqcup f_b \\
\text{sv}(f_2) = 0
\]

Apply \( \text{sv} \) at open superstring amplitude:

\[
\text{sv}(\phi(A^I)) = \left(1 - 2 \sum_{k=1}^{\infty} f_{2k+1} M_{2k+1}\right)^{-1} A_{YM} = \phi(A^{\text{HET}})
\]

Apply \( \phi^{-1} \): \( \text{sv} : \begin{cases} \mathcal{H} \rightarrow \mathcal{H}^{\text{sv}} \\ \zeta^m \rightarrow \zeta^m_{\text{sv}} \end{cases} \)

\[
\zeta^m_{\text{sv}}(2) = 0 \\
\zeta^m_{\text{sv}}(2n + 1) = 2 \zeta^m_{2n+1} \\
\zeta^m_{\text{sv}}(3, 5) = -10 \zeta^m_3 \zeta^m_5
\]

\[
\text{sv}(A^I(\pi)) = A^{\text{HET}}(\pi)
\]

\( J = \text{sv}(Z) \)
Complex vs. iterated integrals

\[ N=5: \]

\[ \int_{z_2, z_3 \in \mathbb{C}} d^2 z_2 \, d^2 z_3 \left( \prod_{i<j}^4 |z_{ij}|^{2s_{ij}} \right) \frac{\prod_{i<j}^4 |z_{ij}|^{2s_{ij}}}{z_{12}z_{23} \overline{z_{12}}\overline{z_{23}}z_{34}} \]

\[ \int_{z_2, z_3 \in \mathbb{C}} d^2 z_2 \, d^2 z_3 \left( \prod_{i<j}^4 |z_{ij}|^{2s_{ij}} \right) \frac{\prod_{i<j}^4 |z_{ij}|^{2s_{ij}}}{z_{12}z_{23} \overline{z_{13}}\overline{z_{32}}z_{24}} \]

\[ = \text{sv} \]

\[ \int_{0<z_2<z_3<1} dz_2 \, dz_3 \left( \prod_{i<j}^4 |z_{ij}|^{s_{ij}} \right) \frac{\prod_{i<j}^4 |z_{ij}|^{s_{ij}}}{z_{12}z_{23}} \]

\[ \int_{0<z_3<z_2<1} dz_2 \, dz_3 \left( \prod_{i<j}^4 |z_{ij}|^{s_{ij}} \right) \frac{\prod_{i<j}^4 |z_{ij}|^{s_{ij}}}{z_{12}z_{23}} \]
Single-valued MZVs

\[ \zeta_{sv}(n_1, \ldots, n_r) \in \mathbb{R} \]

- special class of MZVs, which occurs as the values at unity of SVMPs

polylogarithms: \( \ln(z), \; Li_1(z) = -\ln(1 - z), \; Li_a(z), \; Li_{a_1, \ldots, a_r}(1, \ldots, 1, z) \)

SVMPs: multiple polylogarithms can be combined with their complex conjugates to remove monodromy at \( z = 0, 1, \infty \) rendering the function single-valued on \( \mathbb{P}^1 \backslash \{0, 1, \infty\} \).

\[ \mathcal{L}_2(z) = D(z) = Im \{ Li_2(z) + \ln |z| \ln(1 - z) \} \quad \text{(Bloch-Wigner dilogarithm)} \]

\[ \mathcal{L}_n(z) = Re_n \left\{ \sum_{k=1}^{n} \frac{(-\ln(|z|))^{n-k}}{(n-k)!} Li_k(z) + \frac{\ln^n |z|}{(2n)!} \right\} \quad \text{with:} \quad Re_n = \begin{cases} Im, & n \text{ even} \\ Re, & n \text{ odd} \end{cases} \quad \text{(Zagier)} \]

\[ \mathcal{L}_n(1) = Re_n \{ Li_n(1) \} = \begin{cases} 0, & n \text{ even} \\ \zeta_n, & n \text{ odd} \end{cases} \]
\( \frac{d}{dz} L_{e_0,e_1}(z) = L_{e_0,e_1}(z) \left( \frac{e_0}{z} + \frac{e_1}{1-z} \right) \) with generators \( e_0 \) and \( e_1 \) of the free Lie algebra \( g \).

**Drinfeld associator \( Z \) (generating series of MZVs):**

\[
Z(e_0,e_1) := L_{e_0,e_1}(1) = \sum_{w \in \{e_0,e_1\}^*} \zeta(w) w = 1 + \zeta_2 [e_0,e_1] + \zeta_3 \left( [e_0,[e_0,e_1]] - [e_1,[e_0,e_1]] \right) + \ldots
\]

with the symbol \( w \in \{e_0,e_1\}^* \) denoting a non-commutative word \( w_1w_2\ldots \) in the letters \( w_i \in \{e_0,e_1\} \)

\( \zeta(e_1 e_0^{n_1-1} \ldots e_1 e_0^{n_r-1}) = \zeta_{n_1,\ldots,n_r} \)

\( \zeta(w_1)\zeta(w_2) = \zeta(w_1 \shuffle w_2) \) and \( \zeta(e_0) = 0 = \zeta(e_1) \)

**Deligne associator \( W \) (generating series of SVMZVs):**

Deligne introduced associator \( W \) formally as:

\[
W \circ \sigma Z = Z
\]

with Ihara action \( \circ \) providing formal multiplication rule on group-like formal power series in \( e_0 \) and \( e_1 \)

\[
F(e_0,e_1) \circ G(e_0,e_1) = G(e_0,F(e_0,e_1)e_1 F(e_0,e_1)^{-1}) F(e_0,e_1)
\]

\[
\implies W(e_0,e_1) = \sigma Z(e_0, W e_1 W^{-1})^{-1} Z(e_0,e_1) \quad \text{(definition only uses Ihara action)}
\]

\[
W(e_0,e_1) = \sum_{w \in \{e_0,e_1\}^*} \zeta_{SV}(w) w = 1 + 2 \zeta_3 \left( [e_0,[e_0,e_1]] - [e_1,[e_0,e_1]] \right) + \ldots
\]

F. Brown (2013)
side note: explicit representation of associators in limit \( \text{mod } (g')^2 \)

\[(g')^2 = [g, g]^2 \]

\[u = -\text{ad}_{e_1}, \quad v = \text{ad}_{e_0} \]

\[\text{ad}_x y = [x, y] \]

\[Z(e_0, e_1) = 1 - (uv)^{-1} \left( \frac{\Gamma(1-u) \Gamma(1-v)}{\Gamma(1-u-v)} - 1 \right) [e_0, e_1] \]

relates to open superstring amplitude \( \text{Drummond, Ragoucy (2013)} \)

\[W(e_0, e_1) = 1 + (uv)^{-1} \left( \frac{\Gamma(-u) \Gamma(-v) \Gamma(u+v)}{\Gamma(u) \Gamma(v) \Gamma(-u-v)} + 1 \right) [e_0, e_1] \]

relates to closed superstring amplitude \( \text{St.St. (2013)} \)

\[W \circ \sigma Z = Z \quad \text{KLT relation for associators} \]
Gravitational amplitudes in superstring theory

Type II, type I graviton N-point amplitude:

\[ \mathcal{M} = (-1)^{N-3} \kappa^{N-2} A^t_{YM} S_0 \text{sv}(A^I) \]

only one full-fledged superstring amplitude necessary

with intersection matrix \( S_0 \)

\[ \mathcal{M}_{FT}(1, \ldots, N) = (-1)^{N-3} \kappa^{N-2} \sum_{\sigma \in S_{N-3}} \sum_{\bar{\sigma} \in S_{N-3}} A_{YM}(\sigma) S_0[\sigma|\bar{\sigma}] A_{YM}(\bar{\sigma}) \]

Consequence:

\[ \mathcal{M} = (-1)^{N-3} \kappa^{N-2} A^t_{YM} S_0 A^{\text{HET}} \]

relates gauge and gravitational amplitudes from two different string vacua!

Understand single-valued map sv at the level of string world-sheets
Scalar amplitudes in D=4:

$$\mathcal{A}^{II}(\Phi_{1}^{i_{1}j_{1}}, \Phi_{2}^{i_{2}j_{2}}, \Phi_{3}^{i_{3}j_{3}}, \Phi_{4}^{i_{4}j_{4}}) = \frac{u}{st} \left( t \delta_{1} + s \delta_{2} + \frac{st}{u} \delta_{3} \right)$$

$$\times \left( t \bar{\delta}_{1} + s \bar{\delta}_{2} + \frac{st}{u} \bar{\delta}_{3} \right) \frac{\Gamma(s) \Gamma(u) \Gamma(t)}{\Gamma(-s) \Gamma(-u) \Gamma(-t)}$$

Scalars denote geometric moduli fields describing the internal six-dimensional toroidal $g^{ij}$

$$\delta_{1} = g^{i_{1}i_{2}} g^{i_{3}i_{4}}, \quad \delta_{2} = g^{i_{1}i_{3}} g^{i_{2}i_{4}}, \quad \delta_{3} = g^{i_{1}i_{4}} g^{i_{2}i_{3}},$$

$$\bar{\delta}_{1} = g^{j_{1}j_{2}} g^{j_{3}j_{4}}, \quad \bar{\delta}_{2} = g^{j_{1}j_{3}} g^{j_{2}j_{4}}, \quad \bar{\delta}_{3} = g^{j_{1}j_{4}} g^{j_{2}j_{3}}$$

$$\mathcal{A}^{I}(\Phi_{1}^{j_{1}}, \Phi_{2}^{j_{2}}, \Phi_{3}^{j_{3}}, \Phi_{4}^{j_{4}}) = \left( t \bar{\delta}_{1} + s \bar{\delta}_{2} + \frac{st}{u} \bar{\delta}_{3} \right) \frac{\Gamma(s) \Gamma(1+u)}{\Gamma(1+s+u)}$$

Scalars denote open string moduli fields (D-brane positions or Wilson lines)

$$\mathcal{A}^{II}(\Phi_{1}^{i_{1}j_{1}}, \Phi_{2}^{i_{2}j_{2}}, \Phi_{3}^{i_{3}j_{3}}, \Phi_{4}^{i_{4}j_{4}}) = -\frac{u}{t} \left( t \delta_{1} + s \delta_{2} + \frac{st}{u} \delta_{3} \right)$$

$$\times \text{sv} \left( \mathcal{A}^{I}(\Phi_{1}^{j_{1}}, \Phi_{2}^{j_{2}}, \Phi_{3}^{j_{3}}, \Phi_{4}^{j_{4}}) \right)$$

Similar relations can also be derived for amplitudes involving more than four scalar fields
• By applying naively KLT relations we would not have arrived at these relations

• Much deeper connection between open and closed string amplitudes than what is implied by KLT relations

• Full $\alpha'$-dependence of closed string amplitude is entirely encapsulated by open string amplitude

• Any closed string amplitude can be written as single-valued image of open string amplitude

• Various connections between different amplitudes of different vacua can be established

New kind of duality relating amplitudes involving full tower of massive string excitations (not just BPS states as in most examples of string dualities)
Unity of tree-level field-theory and superstring couplings

Amplitude space = space of physical observables

Unexpected connections between field-theory, open and closed string amplitudes!

Universal $\alpha'$-dependence of all tree-level string amplitudes and their connection to SYM and SUGRA

Amplitudes in non-trivial background, e.g.: warped geometries, AdS$_5$, ...
Concluding remarks

• growing set of interconnections between open & closed amplitudes with gauge theory and supergravity amplitudes

• new kind of duality working beyond usual BPS protected couplings

by combining field and string theory structures obtain information on a possible alternative or dual description of perturbative string amplitudes: obtain amplitudes from first principles
Unified description of superstring and supergravity amplitudes

\[ A_{k,N}^G = \frac{1}{\text{vol}(\text{GL}(2))\text{vol}(\tilde{\text{GL}}(2))} \int d^2\sigma_1 \cdots d^2\sigma_N \int d^2\tilde{\sigma}_1 \cdots d^2\tilde{\sigma}_N H_N(\sigma, \tilde{\sigma}) \]
\[ \times \prod_{\alpha=1}^{k} \delta^4|4| \left( \sum_{i=1}^{N} C_{\alpha i}^V[\sigma] \mathcal{W}_i(\eta) \right) \prod_{\tilde{\alpha}=1}^{k} \delta^4|4| \left( \sum_{i=1}^{N} C_{\tilde{\alpha} i}^V[\tilde{\sigma}] \mathcal{W}_i(\tilde{\eta}) \right) \]

\[ A_{k,N}^S = \frac{1}{\text{vol}(\text{SL}(2))\text{vol}(\text{GL}(2))} \int_{D \subset (\mathbb{P}^1)^N} d^2\sigma_1 \cdots d^2\sigma_N \int d^2\tilde{\sigma}_1 \cdots d^2\tilde{\sigma}_N H_N(\sigma, \tilde{\sigma}) \]
\[ \times \prod_{1 \leq i < j \leq N} |(i,j)|^{s_{ij}} \prod_{\tilde{\alpha}=1}^{k} \delta^4|4| \left( \sum_{i=1}^{N} C_{\tilde{\alpha} i}^V[\tilde{\sigma}] \mathcal{W}_i(\tilde{\eta}) \right) \]

- Striking match between supergravity and open superstring tree-level amplitude communicated by Hodges’ determinant (KLT kernel)
- KLT kernel seems to be the key ingredient in our description

Taylor, St.St. (2013)
Generic N-point string form factor (disk integral):

\[ B_N(\{s_{k,l}\}, \{n_{k,l}\}) = \left( \prod_{i,j \in P} \int_0^\infty du_{i,j} \ u_{i,j}^{s_{i,j}-1+n_{i,j}} \ \theta(1-u_{i,j}) \right) \delta(\{u_{p,q}\}) \]

**with product of** \( \frac{(N-2)(N-3)}{2} \) **delta-functions:**

\[ \delta(\{u_{p,q}\}) = \prod_P' \delta \left( u_P - 1 + \prod_{\tilde{P}} u_{\tilde{P}} \right) \]

**assembled by rules according to a Pascal triangle**

and \( \frac{1}{2}N(N-3) \) **variables** \( u_{i,j} : \quad 0 \leq u_{i,j} \leq 1 \)

\[
\begin{align*}
 i = 2, & \quad j = 3, \ldots, N - 1 , \\
 i = 3, & \quad \ldots, N - 1 < j = 4, \ldots, N
\end{align*}
\]

- Multi-dimensional Mellin transforms from string world-sheet boundary D to the Mellin space of Mandelstam invariants
- Multiple (inverse) Mellin transforms trivialize tree-level string amplitudes
- Mellin transform from string world-sheet into dual space of kinematic invariants thus bypassing space-time, cf. quantum mechanical Fourier transform: \( x^{\alpha'} s \leftrightarrow e^{ikx/\hbar} \)
- Correlation functions from delta-functions and residua integrals