Four point scattering from Amplituhedron

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Nima Arkani-Hamed, JT, 1312.2007
Sebastian Franco, Daniele Galloni, Alberto Mariotti, JT, in progress
JT, in progress
Object of interest

- Scattering amplitudes in planar $\mathcal{N} = 4$ SYM.
- Huge progress in recent years both at weak and strong coupling.
- Generalized unitarity, Twistor string theory, BCFW recursion relations, Leading singularity methods, Relation between amplitudes and Wilson loops, Yangian symmetry, Strong coupling via AdS/CFT, Symbol of amplitudes, Flux tube S-matrix, Positive Grassmannian and Amplituhedron,...
- Planar $\mathcal{N} = 4$ SYM is integrable: It is believed that scattering amplitudes in this theory should be exactly solvable.
- Long list of people involved in these discoveries....
Integrand

- The amplitude $\mathcal{M}_{n,k,L}$ is labeled by three indices: $n$ - number of particles, $k$ - SU(4) R-charge, $L$ is the number of loops.

- Integrand: Well-defined rational function to all loop orders in planar limit: sum of all Feynman diagrams prior to integration.

$$\mathcal{M}_{n,k,L} = \int d^4 \ell_1 d^4 \ell_2 \ldots d^4 \ell_L \mathcal{I}_{n,k,L}$$

- It is completely fixed by its singularities: locality (position of poles) and unitarity (residues on these poles).

- This is an object of our interest: there is a purely geometric definition of this object which does not make any reference to field theory – Amplituhedron.

- There is also a strong evidence of similar structures in the integrated amplitudes.
Different expansion of scattering amplitudes using fully on-shell gauge-invariant objects.

Given by gluing together on-shell 3pt amplitudes.

Explicitly constructed for Yang-Mills theory, and found the expansion of the amplitude in planar $\mathcal{N} = 4$ SYM but these objects exist in any QFT.

On-shell diagrams make the Yangian symmetry of planar $\mathcal{N} = 4$ SYM manifest, not local in space-time.

Direct relation between on-shell diagrams and Positive Grassmannian $G_+(k, n)$. 
Positive Grassmannian and On-shell diagrams

[Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, JT, 1212.5605]

- $G_+(k, n)$: $(k \times n)$ matrix mod $GL(k)$

\[
C = \begin{pmatrix}
  * & * & \ldots & * \\
  \vdots & \vdots & \ldots & \vdots \\
  * & * & \ldots & * 
\end{pmatrix}
\]

where all maximal minors are positive, $(a_{i_1} a_{i_2} \ldots a_{i_k}) > 0$.

- Stratification: cell of $G_+(k, n)$ of dimensionality $d$ given by a set of constraints on consecutive minors.

- For each cell of dimensionality $d$ we can find $d$ positive coordinates $x_i$, and associate a logarithmic form

\[
\Omega_0 = \frac{dx_1}{x_1} \ldots \frac{dx_d}{x_d}
\]

- The particular linear combination of on-shell diagrams (cells of $G_+(k, n)$) is provided by recursion relations.

- Idea: they glue together into a bigger object.
We can define Amplituhedron $A_{n,k,L}$ which is a generalization of positive Grassmannian.

For tree-level $L = 0$, it is a map: $G_+(k, n) \rightarrow G(k, k + 4)$ defined as

$$Y = C \cdot Z \quad \text{where} \quad Z \in M_+(k + 4, n)$$

There is a generalization for the loop integrand which involves new mathematical objects.

In addition we also describe $L$ lines $\mathcal{L}_1, \ldots, \mathcal{L}_L$.

$$C \in G_+(k, n), \quad D_{i_1 \ldots i_m} \in G(k + 2m, n)$$

where $D$ is combination of $C$ and $m$ lines $\mathcal{L}_i$. Then we do the same map,

$$Y = C \cdot Z, \quad \mathcal{Y} = D \cdot Z$$
Amplituhedron

[Arkani-Hamed, JT, 1312.2007]

» The amplitude is then given by the form with logarithmic singularities on the boundaries of this space.

» Logarithmic singularities: if the boundary is characterized by $x = 0$, it is just $\Omega \to \frac{dx}{x}\Omega_0$.

» This is a purely bosonic form but we can extract a supersymmetric amplitude from it: instead of $(Z, \eta)$ we have one $(4 + k)$-dimensional bosonic variables.

» Two ways how to calculate the form:
  » Fix it from the definition (it is unique).
  » Triangulate the space: for each term in the triangulation we have trivial form

$$\Omega = \frac{dx_1}{x_1} \frac{dx_2}{x_2} \ldots \frac{dx_d}{x_d}$$

and we sum all pieces. On-shell diagrams via recursion relations provide a particular triangulation.
Four-point amplitudes

- The number of Feynman diagrams grows extremely rapidly. Natural strategy: find a basis of scalar and tensor integrals.
- The calculation of integrand of 4pt amplitudes has a long history
  - 1-loop: Brink, Green, Schwarz (1982)
  - 2-loop: Bern, Rozowski, Yan (1997)
  - 3-loop: Bern, Dixon, Smirnov (2005)
  - 4-loop: Bern, Czakon, Dixon, Kosower, Smirnov (2006)
  - 5-loop: Bern, Carrasco, Johannson, Kosower (2007)
  - 6,7-loop: Bourjaily, DiRe, Shaikh, Spradlin, Volovich (2011)

- Even in a suitable basis there is a fast growth of the number of diagrams – no sign of simplification.

<table>
<thead>
<tr>
<th>$L$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
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<td># of diagrams</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>8</td>
<td>34</td>
<td>256</td>
<td>2329</td>
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</table>

The 7-loop result is several millions of terms.
Four-point amplitudes

- BDS ansatz [Bern, Dixon, Smirnov, 2005] for the integrated expression for MHV amplitudes in dimensional regularization

\[ M_{n,L} = \exp \left[ \sum_{L=1}^{\infty} \lambda^L \left( f^{(L)}(\epsilon) M_{n,1}(L\epsilon) + C^{(L)} + O(\epsilon) \right) \right] \]

where

\[ f^{(L)}(\epsilon) = f_0^{(L)} + \epsilon f_1^{(L)} + \epsilon^2 f_2^{(L)} \]

- The leading IR divergent piece is given by

\[ f(\lambda) = \sum_{L=1}^{\infty} f_0^{(L)} \lambda^L \]

is known as cusp anomalous dimension, which also governs the scaling of twist-two operators in the limit of large spin \( S \),

\[ \Delta \left( \text{Tr} [Z D^S Z] \right) - S = f(\lambda) \log S + O(S^0) \]

It satisfies BES [Beisert, Eden, Staudacher, 2006] integral equation which can be solved analytically to arbitrary order.
Four-point amplitudes

▶ There is a tension between results for the integrand and the integrated answer.

▶ Integrand is a rational function with infinite complexity for $L \to \infty$ (it must capture all cuts) but the non-trivial part of the integrated result is given by simple functions of coupling.

▶ Important question: Is there a sign of this simplification at the integrand level? What is the role of integrability?

▶ The ultimate goal:

  ▶ Describe the Amplituhedron space for integrand, its stratification and topological properties.
  ▶ Try to find the form with log singularities to all loops (if it exists in a closed form).
  ▶ If yes, try to find a way how to extract (perhaps some natural deformation [Beisert, Broedel, Ferro, Lukowski, Meneghelli, Plefka, Rosso, Staudacher, …]) a BES equation – ie. understand the integration process as some kind of geometric map.
Four-point amplitudes from Amplituhedron

[Arkani-Hamed, JT, 1312.7878]

- The definition of the Amplituhedron in case of four point amplitudes at arbitrary \( L \) is very simple:
  - Let us have \( 4L \) positive parameters,

\[
x_i, y_i, z_i, w_i \geq 0 \quad \text{for } i = 1, 2, \ldots L
\]

which satisfy \( L(L - 1)/2 \) quadratic inequalities.

\[
(x_i - x_j)(w_i - w_j) + (y_i - y_j)(z_i - z_j) \leq 0 \quad \text{for all pairs } i, j
\]

- The amplitude is then the form with logarithmic singularities on the boundaries of this space.

- In this special case the \( Z \)-map is not present and the external data are irrelevant.
One-loop amplitude

- We have four parameters $x_1, y_1, z_1, w_1 \geq 0$
- There is no quadratic condition, the form with logarithmic singularities on the boundaries $(0, \infty)$ is just

$$
\Omega = \frac{dx_1}{x_1} \frac{dy_1}{y_1} \frac{dz_1}{z_1} \frac{dw_1}{w_1}
$$

- We can solve for parameters $x_1, y_1, z_1, w_1$ in terms of kinematical variables

$$
\Omega = \frac{\langle AB \, d^2 Z_A \rangle \langle AB \, d^2 Z_B \rangle \langle 1234 \rangle^2}{\langle AB12 \rangle \langle AB23 \rangle \langle AB34 \rangle \langle AB41 \rangle} = \frac{d^4 \ell \, st}{\ell^2 (\ell + p_1)^2 (\ell + p_1 + p_2)^2 (\ell - p_4)^2}
$$
Two-loop amplitude

- For $L = 2$ we have $x_1, y_1, z_1, w_1, x_2, y_2, z_2, w_2 \geq 0$ which satisfy quadratic relation

$$(x_1 - x_2)(w_1 - w_2) + (y_1 - y_2)(z_1 - z_2) \leq 0$$

- The form has the form

$$\Omega = \frac{dx_1 \, dx_2 \ldots \, dz_2 \, N(x_1, x_2 \ldots z_2)}{x_1 y_1 w_1 z_1 x_2 y_2 w_2 z_2 \left[ (x_1 - x_2)(w_1 - w_2) + (y_1 - y_2)(z_1 - z_2) \right]}$$

It is a 8-form with 9 poles – non-trivial numerator.

- There are two different strategies to find this form:
  - Expand it as a sum of terms with 8 poles with no numerator – triangulation. [Arkani-Hamed, JT, 1312.7878]
  - Fix the numerator directly. [JT, in progress]
Fixing the two-loop amplitude

- Example 1: calculate residuum $y_1 = y_2 = x_2 = 0$,

$$\Omega = \frac{d x_1 \, d z_1 \, d z_2 \, d w_1 \, d w_2 \, \tilde{N}}{x_2^2 w_1 z_1 w_2 z_2 (w_1 - w_2)} \rightarrow \tilde{N} \sim x_1$$

- Example 2: For $x_2 = w_2 = y_2 = z_2 = 0$ we have

$$x_1 w_1 + y_1 z_1 \leq 0$$

and therefore the numerator must vanish $\tilde{N} = 0$.

- These conditions fix completely the numerator up to overall constant to be

$$N = x_1 w_2 + x_2 w_1 + y_1 z_2 + y_2 z_1$$
Topology of Amplituhedron

[Franco, Galloni, Mariotti, JT, in progress]

- Topology of $G_+ (k, n)$: Euler characteristic $= 1$, it is a very non-trivial property of the space.

- The $L = 1$ case is just $G_+ (2, 4)$

<table>
<thead>
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<th>2</th>
<th>1</th>
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<tbody>
<tr>
<td># of boundaries</td>
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<td>4</td>
<td>10</td>
<td>12</td>
<td>6</td>
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with Euler characteristic $\mathcal{E} = 1$.

- We can count boundaries of $L = 2$,

<table>
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<th>dim</th>
<th>8</th>
<th>7</th>
<th>6</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td># of boundaries</td>
<td>1</td>
<td>9</td>
<td>44</td>
<td>144</td>
<td>286</td>
<td>340</td>
<td>266</td>
<td>136</td>
<td>34</td>
</tr>
</tbody>
</table>

Alternating sum of these numbers gives $\mathcal{E} = 2$.

- There are preliminary results for $L = 3, 4$ which show similar topological properties.

- The non-trivial topology is probably closely related to the complexity of the logarithmic form.
Amplitude at $L$-loops

- At $L$-loops we have $4L$ positive parameters $x_i, y_i, z_i, w_i \geq 0$ with

\[(x_i - x_j)(w_i - w_j) + (y_i - y_j)(z_i - z_j) \leq 0 \quad \text{for all pairs } i, j\]

- We can write the general form

\[
\Omega = \frac{dx_1 \, dy_1 \ldots \, dz_L \, N(x_1, \ldots, z_L)}{x_1y_1w_1z_1 \ldots x_Ly_Lw_Lz_L \prod_{i,j} [(x_i - x_j)(w_i - w_j) + (y_i - y_j)(z_i - z_j)]}
\]

and fix the numerator from constraints. So far this is too hard to solve in general.

- There are two types of special cases we can solve at this moment:
  - Find the form on certain residues (cuts of the amplitude).
  - Smaller set of positivity conditions.
Cuts of $L$-loop amplitude

[Arkani-Hamed, JT, 1312.7878]

- There are certain residues of $\Omega$ (cuts of the amplitude) we can solve for all $L$.
- Example: quadratic equations factorize, $z_i = 0$.

$$(x_i - x_j)(y_i - y_j) \leq 0$$

- All $x_i, y_j$ are then ordered. The form is

$$\Omega = \frac{1}{w_1 \ldots w_L} \sum_\sigma \Omega_\sigma$$

where

$$\Omega_{1\ldots n} = \frac{1}{x_1(x_1 - x_2) \ldots (x_{L-1} - x_L)y_L(y_L - y_{L-1}) \ldots (y_2 - y_1)}$$
Toy model for $L$-loop amplitude

[JT, in progress]

- Let us consider a reduced version of our problem: $4L$ positive variables $x_i, y_i, z_i, w_i \geq 0$, ordered $i = 1, 2, \ldots n$.
- We impose quadratic conditions only between adjacent indices

$$
(x_i - x_{i+1})(w_i - w_{i+1}) + (y_i - y_{i+1})(z_i - z_{i+1}) \leq 0
$$

- The form is then

$$
\Omega = \frac{dx_1 \, dy_1 \ldots \, dz_L \, N_L(x_1, \ldots, z_L)}{x_1 y_1 w_1 z_1 \ldots x_L y_L w_L z_L \prod_{j=i+1}^L [(x_i - x_j)(w_i - w_j) + (y_i - y_j)(z_i - z_j)]}
$$

We can fully constrain the numerator $N_L$ and write down the explicit solution for any $L$.

- The solution has an interesting structure:

$$
N_L = N_2(12)N_2(23)\ldots N_2(L1) + \Delta_L
$$

where $N_2$ is the $L = 2$ numerator.
Conclusion

- The problem of calculating the integrand of four-point amplitudes in planar $\mathcal{N} = 4$ SYM can be reformulated in the context of the Amplituhedron.

- We can easily define the problem but to find the solution to all loop orders is hard. I showed some partial results but the complete solution is still missing.

- There must be a close relation between the topology of the space and non-triviality of the form.

- Four point amplitudes as an ideal test case: if the full perturbative expansion for amplitudes can be solved exactly (despite there is no evidence for it so far) we should see it here.