PHYSICS 103 THIRD HOUR EXAM

Instructions: When you are told to begin, check that this examination booklet contains all the numbered pages from 3 through 13.

Princeton University Undergraduate Honor Committee

This examination is administered under the Princeton University Honor Code. Students should sit one seat apart from each other, if possible, and refrain from talking to other students during the exam. All suspected violations of the Honor Code must be reported to the Honor Committee Chair at honor@princeton.edu.

The checked items below are permitted for use on this examination. Any item that is not checked may not be used and should not be in your working space. Assume items not on this list are not allowed for use on this examination. Please place items you will not need out of view in your bag or under your working space at this time. University policy does not allow the use of electronic devices such as cell phones, PDAs, laptops, MP3 players, iPods, etc. during examinations. Students may not wear headphones during an examination.

☐ Course textbooks  ☐ Course Notes  ☐ Other printed materials

NOTHING IS ALLOWED EXCEPT THIS EXAMINATION, A PEN OR PENCIL, AND YOUR CALCULATOR. THE CALCULATOR POLICY (SEE NEXT PAGE) IS IN PLACE AS ALWAYS.

Students may only leave the examination room for a very brief period without the explicit permission of the instructor. The exam may not be taken outside of the examination room.

This is a timed examination. You will have 50 minutes to complete this exam. During the examination, the Professor or a preceptor will be outside the exam room.

Rewrite and sign the pledge: I pledge my honor that I have not violated the Honor Code during this examination.

Signature
FURTHER INSTRUCTIONS

The exam contains FOUR problems with multiple parts, worth varying numbers of points, which total to 100.

Do not panic or be discouraged if you cannot do every part of every problem; there are both easy and hard parts in this exam. If a part of a problem depends on a previous answer you have not obtained, assume it and proceed. Keep moving and finish as much as you can!

Read each problem carefully. You must show your work — the grade you get depends on how well the grader can understand your solution even when you write down the correct answer. Always write down analytic answers first and only then calculate numerical values if requested. Check that the dimensions of your answers are correct.

Write your final answers in the boxes provided.

To refresh your memory, here is the calculator policy from the Blackboard website.

**Calculator Policy**: Calculators are tools used by all physicists. Many of these devices have advanced features that are very useful. However, not all students have calculators with these features, and our rules are designed for fairness. On tests and the final: No advanced features may be used at any time. Calculator use is restricted to arithmetic and evaluation of trig functions, exponentials, logs, square roots, etc. You are not allowed to use, for example,

- calculator memory for storage of physics formulas or constants
- automatic equations solving
- graphing features
- functions such as quadratic formula

You are also not allowed to use any of these features for checking answers. Violation of these rules is a violation of the Honor Code.

**DO ALL THE WORK YOU WANT TO HAVE GRADED IN THIS EXAMINATION BOOKLET! YOU WILL NOT BE ALLOWED TO HAND IN ANYTHING ELSE.**
A space docking station of the future may be modeled as a uniform rod of length $L$ and mass $M$. A space shuttle, modeled as a point mass $2M$ docks a distance $L/4$ from one end, as indicated in the figure.

a) (5 pts) Where is the center of mass of this system (station-plus-shuttle)? Consider the origin of the $x$-axis to be at the left end of the space station, as sketched in the figure. Remember to show your work for full credit.

\[
X_{cm} = \frac{1}{3M} \left[ \frac{ML}{2} + 2M \frac{3L}{4} \right]
\]

\[
X_{cm} = \left( \frac{1}{6} + \frac{1}{2} \right) L
\]

\[
x_{cm} = \frac{2}{3} L
\]
b) (15 pts) What is the moment of inertia, $I_{cm}$, of this system about an axis through the center of mass and orthogonal to the plane of the figure? Show all of your work, including an integral, for full credit.

\[
I_{cm} = I_{cm}^{rod} + I_{cm}^{shuttle} \\
I_{cm}^{rod} = \int -z^2 \, dm = \frac{m}{L} \int_{-\frac{2}{3}L}^{\frac{1}{3}L} x^2 \, dx = \frac{m}{L} \left[ \frac{1}{3} x^3 \right]_{-\frac{2}{3}L}^{\frac{1}{3}L} = \frac{1}{3} mL^2
\]

\[
I_{cm}^{rod} = \frac{m}{L} \left[ \frac{1}{3} \left( \frac{L^3}{27} - \frac{1}{3} \left( -\frac{2}{3} \right)^3 \right) \frac{1}{2} \right] = \frac{1}{9} mL^2
\]

\[
I_{cm}^{shuttle} = 2 m \bar{x}^2 = 2m \left( \frac{3}{4} L - \frac{2}{3} L \right)^2 = \frac{1}{72} mL^2
\]

\[
I_{cm} = \frac{1}{72} mL^2 + \frac{1}{9} mL^2 = \frac{1}{8} mL^2
\]

You can also do it by using the parallel axis theorem.
c) (10 pts) Consider the docking process. The space station is at rest. The space shuttle arrives at speed $v_0$ along the trajectory shown in the figure above and collides and sticks to the station at the docking point. What is the angular frequency of the station-plus-shuttle system after the collision? (If you did not find an answer to part b), you may use $I_{cm} = ML^2/12$ for this part.)

Angular momentum of the combined system (station + shuttle) is conserved. (The system is isolated)

\[ L_{	ext{before}} = L_{	ext{after}} \]

\[ m_{	ext{cm}} v_0 L = \frac{1}{2} m L^2 \omega_f \]

\[ L_{	ext{before}} = 2m v_0 \left( \frac{3L}{4} - \frac{2L}{3} \right) = \frac{m v_0 L}{6} \]

\[ |L_{	ext{after}}| = I_{cm} \omega_f = \frac{1}{2} m L^2 \omega_f \]

After the collision, the system will rotate about the center of mass.
Problem 2. Orbits (20 pts)

A satellite in geostationary orbit moves in a circular orbit with a period of $T_d = 24$ hrs, so that it appears to remain stationary over a point on the Earth's equator. (This is a special case of the geosynchronous orbit.) The Earth has radius $R_E = 6.4 \times 10^6$ m and mass $M_E = 6.0 \times 10^{24}$ kg, and Newton's gravitational constant is $G = 6.7 \times 10^{-11} \text{m}^3\text{kg}^{-1}\text{s}^{-2}$.

a) (5 pts) What is the linear speed $v$ of the satellite in terms of $T_d$ and $r$?

$$\boldsymbol{v} = \omega \boldsymbol{r} = \frac{2 \pi}{T} \boldsymbol{r} \quad \text{(5 pts)}$$

b) (5 pts) Write down Newton's third law for the satellite to find an equation relating $r$, $v$, $G$ and $M_E$.

$$|\vec{F}_{\text{sat, earth}}| = \frac{G M_e m}{r^2} \quad \text{(5 pts)}$$

$$\vec{F}_{\text{sat, earth}} = \frac{G M_e m}{r^2} \quad \text{(5 pts)}$$

$$G \frac{m \ddot{r}}{r} = m \frac{v^2}{r} \quad \text{(5 pts)}$$

$$\frac{\frac{v^2}{r}}{2} = \frac{G M_e}{2r} \quad \text{(5 pts)}$$

$$\frac{v^2}{2} = \frac{G M_e}{r}$$

$$G M_e m = \left(\frac{v}{2}\right)^2$$
c) (10 pts) At what radius \( r \) from the center of the Earth does a geostationary satellite orbit? Show all of your work for full credit. First give an analytical expression (in terms of \( G, T_d, R_E, M_E \), as needed) and then a numerical one.

\[
\frac{v^2}{r} = \frac{GM_E}{r^2}
\]

From part a) \( v = \frac{2\pi r}{T_d} \) \( \Rightarrow \) \( \left(\frac{2\pi r}{T_d}\right)^2 = \frac{GM_E}{r^2} \)

\[
2^3 = \frac{GM_E T_d^2}{(2\pi)^2}
\]

\[
z = \left(\frac{GM_E T_d^2}{4\pi^2}\right)^{\frac{1}{3}}
\]

(analytic) \( r = \sqrt[3]{\frac{GM_E T_d^2}{4\pi^2}} \)

(numerical) \( r = 4.23 \times 10^7 \text{ m} \)
An essentially massless spring with spring constant $k$ is standing upright on a table with its bottom end fastened to the table and a massless tray attached to its top. A block of mass $m$ is dropped from a distance $h$ above the top of the spring. The block has sticky tape on its bottom so it sticks to the tray and oscillates in simple harmonic motion subsequent to its landing.

a) (5 pts) What is the period $T$ of the resulting motion?

$$\omega = \sqrt{\frac{k}{m}}$$

$$T = 2\pi \sqrt{\frac{m}{k}}$$
b) (5 pts) What is the maximum compression of the spring during the resulting motion? Express your answer in terms of $h$, $m$, $k$, and $g$ as needed.

\[
\begin{align*}
\text{Initial gravity potential energy} & \quad \text{compared to final.} \\
V_i &= mg(h + l_{\text{max}}) \\
\text{Final spring potential energy} & \quad V_f = \frac{1}{2} k l_{\text{max}}^2 \\
\text{set equal:} & \\
\frac{1}{2} k l_{\text{max}}^2 &= mg(h + l_{\text{max}}) \\
l_{\text{max}}^2 &= \frac{2mg}{k} l_{\text{max}} - \frac{2mg}{k} \cdot h = 0 \\
l_{\text{max}} &= \left(\frac{2mg}{k} \pm \sqrt{\frac{4m^2g^2}{k^2} + 4 \cdot \frac{2mg}{k} h}\right) / 2 = \frac{mg}{k} \left(1 \pm \sqrt{1 + \frac{2hl}{mg}}\right)
\end{align*}
\]

Maximum compression = 
\[
\ell_0 \left(1 + \sqrt{\frac{2h}{\ell_0}}\right)
\]

There are two solutions but only one of them is physical.

To see which one you have recognize that when spring and the mass one in equilibrium (i.e. not oscillating)

\[
k\Delta l = mg \quad \Delta l = \frac{mg}{k}
\]

So only 
\[
l_{\text{max}} = \ell_0 \left(1 + \sqrt{2 \frac{h}{\ell_0}}\right)
\]

is valid.
c) (15 pts) Find an expression for the amplitude $A$ of the resulting oscillations in terms of $h$, $m$, $k$ and $g$ as needed. You must make every step of your work clear for full credit.

\[ A = l_{\text{max}} - l_0 \]

\[ = l_0 \left( 1 + \sqrt{1 + \frac{2h}{l_0}} \right) - l_0 \]

\[ A = l_0 \sqrt{1 + \frac{2h}{l_0}} \]

\[ l_0 = \frac{mg}{k} \]
Problem 4. Conical Pendulum and the Vector Nature of $L$ (25 pts)

A ball of mass $m$ is attached by a massless string to a frictionless pivot which sits at the origin of an $xyz$ coordinate system. The $z$ axis is vertical. The ball rotates with angular frequency $\omega$, describing a circle of radius $R$ in a horizontal plane a distance $h$ below the pivot. At time $t = 0$ the ball is at $\left(x, y, z\right) = \left(R, 0, -h\right)$.

a) (5 pts) At $t = 0$, what is the torque $\tau$ on the ball, about the pivot point? Express your answer as a vector in the $xyz$ coordinate system, in terms of $R$, $h$, $\omega$, $m$ and $g$ as needed.

\[ \vec{\tau} = \vec{r} \times \vec{F} \]

\[
\begin{cases} 
\vec{F} = R \hat{\imath} - h \hat{k} \\
\vec{\tau} = \omega g R \hat{\jmath} \end{cases}
\Rightarrow \vec{\tau} = \left(-mgR\right)\left(\hat{\imath} \times \hat{\jmath}\right)
\]

\[ \vec{\tau} = m g R \hat{\jmath} \]

\[ \tau = m g R \hat{\jmath} \]
b) (5 pts) At \( t = 0 \), what is the angular momentum \( \vec{L} \) of the ball relative to the pivot point? Express your answer as a vector in the \( xyz \) coordinate system, in terms of \( R \), \( h \), \( \omega \), \( m \) and \( g \) as needed.

\[
\vec{L} = \vec{r} \times \vec{p} = \left\{ \begin{array}{l}
\vec{r}(0) = R \hat{x} - h \hat{z} \\
\vec{p}(0) = m v \hat{y} = m \omega R \hat{y}
\end{array} \right.
\]

\[
\vec{L}(0) = m \omega R^2 (\hat{x} \times \hat{y}) - m \omega R h (\hat{z} \times \hat{y})
\]

\[
\vec{L}(0) = m \omega R^2 \hat{z} + m \omega R h \hat{x}
\]

\[
\vec{L} = m \omega R h \hat{x} + m \omega R^2 \hat{z}
\]

---

c) (5 pts) What are the components of the position vector of the ball as a function of time, \( x(t) \), \( y(t) \), and \( z(t) \)?

\[
x(t) = R \cos \omega t \\
y(t) = R \sin \omega t \\
z(t) = -h
\]

---

- \( z = -h \) plane 
  \( \Rightarrow \)

- \( \vec{r}(t) \)
  - \( x(t) = R \cos \omega t \)
  - \( y(t) = R \sin \omega t \)
  - \( z(t) = -h \) --- motion in
  - \( -h \) plane
d) (5 pts) Find the angular momentum vector as a function of time, \( L(t) \).

\[
\vec{L}(t) = \vec{r}(t) \times \vec{p}(t)
\]

\[
\vec{r}(t) = R \cos \omega t \hat{x} + R \sin \omega t \hat{y} - h \hat{z}
\]

\[
\vec{p}(t) = m \omega R (- \sin \omega t \hat{x}, \cos \omega t \hat{y})
\]

\[
\vec{v}(t) = (\omega R \sin \omega t, \omega R \cos \omega t, 0)
\]

Hence

\[
\vec{L}(t) = m \omega R h \cos \omega t \hat{x} + m \omega R h \sin \omega t \hat{y} + m \omega R^2 \hat{z}
\]

\[
L = (m \omega R h \cos \omega t, m \omega R h \sin \omega t, m \omega R^2)
\]

e) (5 pts) Are any components of the angular momentum vector conserved (constant)? If so, which? Note: you can answer this even if you have not worked out the previous part exactly.

\[
\vec{E} = \frac{d\vec{L}}{dt}
\]

\[
\frac{dL_z}{dt} = 0 \quad \text{at any time,}
\]

Therefore \( \frac{dL_z}{dt} = 0 \implies L_z = \text{constant} \).

The \( L_z \)-component is conserved!

This can also be inferred from the solution of item d/.