SYMmetry Breaking in Melonic Models

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Abstract

This thesis discusses novel symmetry breaking phenomena observed in models with a melonic large $N$ limit, such as Sachdev-Ye-Kitaev (SYK) models and tensor models. In such a limit, these models are often nearly conformal. However, this thesis provides examples where the would-be nearly conformal points violate unitarity and are, therefore, not realized as the physical solutions of the models. In those cases, the physical solutions may spontaneously break certain symmetries.

Chapter 2 is based on work with Jaewon Kim, Igor R. Klebanov and Grigory Tarnopolsky [1]. It is devoted to the study of a system consisting of two Majorana tensor models coupled through a quartic tetrahedral interaction. It shows that, for certain range of the coupling, the nearly conformal solution becomes non-unitary and unstable. It identifies the operator whose scaling dimension goes complex, and shows through numerical methods that at low temperature, such an operator acquires a vacuum expectation value (VEV), which spontaneously breaks a $Z_2$ particle hole symmetry.

Chapter 3 is based on work with Igor R. Klebanov, Alexey Milekhin, and Grigory Tarnopolsky [2]. It extends the work of chapter 2 to the spontaneous breaking of a $U(1)$ symmetry. In particular, Goldstone modes are identified. Specifically, the chapter contains the study of a system consisting of two copies of the complex Sachdev-Ye-Kitaev (SYK) or tensor models coupled by a quartic term preserving the $U(1) \times U(1)$ symmetry. It reveals that the scaling dimension of certain bilinear operator charged under axial $U(1)$ becomes complex for some range of the quartic coupling, and at low temperature such an operator condenses, thus spontaneously breaks $U(1)$ symmetry.

Chapter 4 is based on work with Matthew Heydeman and Gustavo Joaquin Turiaci. It is devoted to the study of $\mathcal{N} = 2$ supersymmetric SYK models with complex fermions, at non-zero charge. A new $\mathcal{N} = 2$ SYK model with multiple $U(1)$ symmetries is also proposed. In both models, a nearly conformal solution emerges at low temperatures. For a critical charge a high entropy to low entropy phase transition is found through numerical methods.
In the new model proposed, such a low entropy phase spontaneously breaks the $N = 2$ supersymmetry.
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Chapter 1

Introduction

In the quest for the fundamental laws of nature, good toy models are especially valuable. They are deliberately simplified, but they can still capture essential aspects of the physical phenomena to be explained. Compared to the original problem, they are usually more tractable and easier to solve. For example, in the Ising model, a set of discrete variables on a lattice with only nearest neighbor interactions is enough to capture the physics of ferromagnets [3]. In fact, the Ising models in one and two dimensions are exactly solvable [4]. Another example is the QCD with a large number of colors [5]. When such a theory in two dimensional spacetime is considered, the meson spectrum becomes more easily solvable [6]. Some toy models are even interesting in their own right and unveil rich mathematical or physical structures otherwise undiscovered. For example, a class of toy models for quantum field theories called topological field theories can be very useful to understand the topological structure of 3-manifolds, and through them new topological invariants are discovered [7]. To some extent, many examples considered in String Theory, such as the Anti-de-Sitter/Conformal field theory(AdS/CFT) duality [8–10], are toy versions of the world of our own. All such toy models help us give an explanation to the essence of some physical phenomenon, and in many cases provide us a handle to a much more difficult problem.

A class of particularly powerful toy models involves taking the number of degrees of
freedom to be large (large $N$), and considering a formal asymptotic expansion of the theory at infinite degrees of freedom. Renowned examples of vector, matrix models and more recently, tensor models are all of this kind (see [11,12] for reviews). At large $N$, these theories become much simpler to solve, and yield important physical insights to these strongly coupled quantum systems. Besides, some of the large $N$ quantum field theories can in fact aid us discover more about gravitational physics, through the AdS/CFT correspondence. In that context, the large $N$ limit is a particularly important limit, as on the gravitational side (bulk), it corresponds to a semi-classical limit, where quantum gravity can be approximately studied perturbatively.

The most useful place for toy models is where the underlying physical laws are universal and insensitive to the specifics and microscopic properties of the individual system. In this case it is easier for toy models to capture the essential features. For example, in second order phase transitions, where under the thermodynamics limit, the transitions in the same universal class are described by a single conformal field theory no matter whether it is about water to gas or ferromagnetic to paramagnetic. Conformal field theories in those cases are powerful toy models that in fact precisely describe the actual physics. Solving these 'toy models' would help us unfold crucial features of these physical processes.

Among all the problems that exhibit some degrees of universality, black holes remain the most challenging. They are classical solutions predicted by general relativity, and are surprisingly simple. Unlike a star or a planet, a black hole is simply described by its mass,
angular momentum and charge. However, black holes are also thermal objects that potentially carry a large amount of entropy, proportional to their areas. In particular, for charged or rotating black holes, they can remain large as their temperatures go to zero, hence hinting for potentially a large amount of zero temperature entropy. One of the central questions in theoretical physics is to understand the origin of these microscopic states. Yet such answers are not universal, as the precise microstates require full access to the quantum gravity Hilbert space, and highly depend on the specifics of the gravitational theories. What turns out to be universal is the thermodynamics near zero temperature. It involves the breaking of certain symmetries in a very subtle way.

The Sachdev-Ye-Kitaev (SYK) models are a family of simple toy models for the thermodynamics of the near extremal black holes [13, 14]. They become exactly solvable when the degrees of freedom become large and the couplings become strong. In such a limit, conformal symmetries emerge, but are spontaneously and explicitly broken by the large $N$ saddle. The way they break is exactly identical to how a small temperature breaks the near horizon symmetries of an extremal black hole. As such, SYK models are ideal toy models for black holes. Since they are merely quantum mechanical systems, they can be more easily studied numerically compared to higher dimensional conformal field theories. They may be the simplest model of holography up to date.

However, from a quantum field theory’s perspective, SYK models are unconventional because they involve random interactions. A slightly different toy model, called tensor model, encapsulates the same features of SYK models without using random hamiltonian [15–17]. It involves taking fermions in the tensor representation of the $O(N)$ symmetry group, and has a novel large $N$ limit. Similar to the SYK model, it also exhibits rich physics. Both SYK and tensor models are governed by a class of diagrams called melonic diagrams (see figure 1.4), and sometimes are referred as melonic models.

One of the curious features of such toy models is the instability observed in some variants (see for example, [12]). In these cases, some scalar operator in the conformal solution
acquires a complex scaling dimension. This corresponds to turning on a bulk scalar with mass below its Breitenlohner-Freedman bound, hinting instability in the bulk.

In this thesis, we explore the resolutions to these instabilities— they lead to the spontaneous breaking of certain symmetries in these large $N$ quantum mechanical systems. The symmetry broken phase still carries large entropy as the unbroken phase, hinting for a large horizon in the bulk. We also discover a new kind of instability where certain fermionic operator violates unitarity. In those cases, a high to low entropy transition occurs. It suggests that in the bulk the horizon completely disappears.

In the remaining of the section, we will review the basics of SYK and tensor models, and the black hole physics they describe. We will first give a short review the thermodynamics of nearly extremal black holes in Section 1.1, focusing on features captured by the toy models. In Section 1.2 we will review SYK models and the nearly conformal solutions. In Section 1.3, we give an overview of the thesis.

## 1.1 Near extremal black holes

In this section we will review the basic facts about near extremal black holes that make SYK models relevant to their thermodynamics.

To begin with, we review the how classical thermodynamics of black holes breaks down at low temperature. For our purpose of illustration, it is most convenient to consider a magnetically charged black hole in 4 dimension, whose metric is given by

$$ds^2 = -\frac{(r-r^+)(r-r^-)}{r^2}dt^2 + \frac{r^2}{(r-r^+)(r-r^-)}dr^2 + r^2 d\Omega_2^2, \quad F = Q \sin \theta d\theta \wedge d\phi.$$  \hspace{1cm} (1.1.1)

with magnetic field supported on $S^2$. We choose the magnetically charged solution instead of electrically charged, so that when reducing to AdS$_2$, we would only get extra scalars instead of electric field. As a side, the electrically charged solution is very relevant to the complex

\footnote{Due to large $N$, these systems avoid the no-go theorems on spontaneous symmetry breaking.}
SYK model and models we discuss in chapter 4.

The mass and charge of the black hole are given in terms of the locations of the two horizons:

\[ Q = \sqrt{r_+ r_-}, \quad M = \frac{r_+ + r_-}{2}. \]  

(1.1.2)

We observe the first law of thermodynamics

\[ dM = \frac{r_+^2 - r_-^2}{2r_+^2} d \left( \frac{4\pi r_+^2}{4} \right) + \frac{2\sqrt{r_+ r_-}}{r_+} dQ = T dS + \Phi dQ, \]  

(1.1.3)

and thus identify the temperature to be

\[ T_{BH} = \frac{r_+^2 - r_-^2}{2r_+^2}. \]  

(1.1.4)

Note it’s convenient to define the energy of the black hole,

\[ E = M - Q. \]  

(1.1.5)

The limit \( E \to 0 \) is the same as the zero temperature limit of the black hole. At \( E = 0 \), the black hole is called extremal. Consider an expansion near \( T_{BH} = 0 \). The energy as function of the temperature behaves as

\[ E = 2\pi^2 Q^3 T_{BH}^2 + 16\pi^3 Q^4 T_{BH}^3 + O(T_{BH}^4). \]  

(1.1.6)

Therefore at small \( T_{BH} \), the energy depends on the temperature quadratically. That in fact poses a puzzle because at some point the energy becomes discrete, given by units of Hawking quanta. A single unit of Hawking quanta carries energy proportional to the black hole temperature

\[ E_{\text{Hawking}} = c T_H. \]  

(1.1.7)

At some low temperature the quadratic curve given by (1.1.6) must intercept with the linear
curve of the Hawking quanta. At that temperature the emission of a single Hawking quanta would drastically change the black hole temperature, and therefore the semi-classical analysis breaks down. Such a scale is usually referred as the gap scale, or $E_{\text{gap}}$ of a near extremal black hole, given by

$$E_{\text{gap}} \sim \frac{1}{Q^3}. \quad (1.1.8)$$

However, as we shall see, the use of this terminology is not accurate as not all black holes have a gapped spectrum at this scale. In particular, in [18], it’s shown that the non-supersymmetric black holes do not have such a gap. Yet for supersymmetric black holes in flat space, there is always a gap [19], and that is consistent with considerations from examples in String Theory [20].

To understand the low temperature spectrum of black holes, we consider the near horizon geometry of the near extremal black holes. At extremality, $r_+ = r_-$, we let $z = \frac{r^2}{r_* - r_+}$, and in terms of $z$ near $r - r_+$, the metric (1.1.1) becomes

$$ds^2 \to r_+^2 \left( \frac{1}{z^2} (-dt^2 + dz^2) + d\Omega^2 \right), \quad (1.1.9)$$

which is the geometry of $\text{AdS}_2 \times S^2$. Near extremality, the geometry is approximately $\text{AdS}_2 \times S^2$, but a non-zero temperature explicitly breaks the isometry of $\text{AdS}_2$. In fact the asymptotic symmetries of $\text{AdS}_2$ are generated by the reparametrization

$$t \to f(t), \quad z \to zf'(t). \quad (1.1.10)$$

The transformation (1.1.10) keeps the metric invariant near $z = 0$. However the reparametrization symmetry is spontaneously broken by $\text{AdS}_2$ as a particular boundary time is chosen by the metric. It was shown in [21] that the effect of temperature can be understood as a cut-off imposed at a finite distance. The boundary curve that cuts off the $\text{AdS}_2$ spacetime explicitly breaks the asymptotic reparametrization symmetry. We will see in section 1.2
that SYK model also has an emergent reparametrization symmetry that is spontaneously and explicitly broken.

A more precise analysis requires dimensionally reducing the 4d gravitational theory to 2d, and analyzing the contributions from various Kaluza–Klein modes. In [18], it is shown that we can reliably compute the temperature dependence of the black hole partition function using Jackiw-Teitelboim (JT) gravity. For black holes with general charge \( Q \) and angular momentum \( J \), the effective action is given by

\[
S = \beta M_0(Q, J) - S_0(Q, J) - \frac{1}{4} \int_M d^2 x \sqrt{g} \left( \phi(R + \frac{2}{L^2}) \right) - \frac{1}{2} \int_{\partial M} dt \sqrt{h} \frac{\phi_b Q}{\epsilon} \left( K - \frac{1}{L^2} \right),
\]

(1.1.11)

where \( M_0(Q, J) \) is an effective shift of energy by integrating out the field outside the near horizon region, \( S_0(Q, J) \) is the extremal, and the remaining action describes the JT gravity, where \( R \) and \( K \) are respectively Ricci scalar and extrinsic curvature. In JT gravity, bulk has constant negative curvature, and the bulk part of the on shell action vanishes. The boundary metric is taken to be a constant

\[ h_{tt} = \frac{1}{\epsilon^2}. \]

(1.1.12)

Under the reparametrization (1.1.10) the extrinsic curvature becomes

\[
K = 1 + \epsilon^2 \text{Sch}(f(t), t) + \mathcal{O}(\epsilon^4), \quad \text{Sch}(f(t), t) = \frac{f'''(t)}{f'(t)} - \frac{3}{2} \frac{f''(t)^2}{f'(t)^2}.
\]

(1.1.13)

Thus the two dimension gravitational theory has an effective boundary action given by the Schwarzian derivative. At fixed charge sector the partition function can be written as

\[
Z_{BH} = e^{S_0 - \beta M_0} \int \frac{Df(t)}{(\text{SL}(2, \mathbb{R}))} \exp \left( \frac{2\pi \phi_b}{\beta} \int_0^{2\pi} \text{Sch}(f(t), t) \right).
\]

(1.1.14)

The partition function can be computed in terms of a Schwarzian partition function as well as summing over appropriate charges and their degeneracy. When temperature approaches the gap scale, the \( \phi_b \ll \beta \) and the Schwarzian theory becomes strongly coupled.
A naive saddle point expansion would fail. However, the Schwarzian partition function can be computed exactly at all temperature through a technique called fermionic localization. In fact, the partition function is one-loop exact [22]. This allows a reliable computation for temperature around the gap scale (1.1.8). A continuum of states is observed, showing there is no gap around scale $1/Q^3$. At low temperature, the energy can be worked out

$$E = M_0 + \frac{3}{2\beta} + \frac{2\pi^2 \phi_b}{\beta^2} + \ldots$$  \hspace{1cm} (1.1.15)

Note the linear term in temperature is exactly due to the Schwarzian and this resolves the puzzle due to $E_{\text{gap}}$, as the energy is always higher than the temperature. The resolution is only made possible as the infrared dynamics is governed by the Schwarzian theory which is exactly solvable.

We will see this in Section 1.2 and that the low energy spectrum of the SYK model is exactly determined by the Schwarzian theory. Moreover, the conformal symmetry in SYK breaks in the same way as the asymptotic reparametrization symmetry is broken in AdS$_2$. Such features make SYK model a toy model for near extremal black holes.

1.2 SYK and tensor models as nearly conformal theories

During the past several years there has been a flurry of activity on fermionic quantum mechanical models which are exactly solvable in the large $N$ limit because they are dominated by the so-called melonic Feynman diagrams. Work in this direction began with the Sachdev-Ye-Kitaev (SYK) models [23–26], which have random couplings. More recently, the tensor quantum mechanical models [15, 16], which have continuous symmetry groups and no randomness, were constructed following the body of research on melonic large $N$ tensor models in $d = 0$ [27–33] (for reviews, see [12, 34–36]).
In this section we will introduce the basics of SYK models and its non-random cousin tensor models. We will explain in what sense they are nearly conformal. Before we introduce the models, we first comment on why an exactly conformal theory is uninteresting in one dimension. For a more detailed introduction see [12, 14, 37].

**Nearly conformal theories in one dimension**

Conformal field theories are quantum field theories with conformal invariance. In particular, the conformal symmetries become large in lower dimensions. In two dimensions they become Virasoro symmetry and in one dimension they become the full diffeomorphism symmetry. Thus for a quantum mechanical system, possessing the full conformal symmetry means the loss any possible dynamics, as the only diffeomorphism invariant hamiltonian is $H = 0$.

Another way to see this is to consider the density of states. For a true conformally invariant theory, there can not be a scale parameter, and thus by dimensional analysis, there can only be two choices for the density of states:

\[ \rho(E) = e^{S_0} \delta(E) \text{ or } \frac{1}{E}. \]  

The former is conjectured to be the case of supersymemtric black holes, where $e^{S_0}$ consists of BPS states which are separated from the rest of the spectrum by a gap. The latter, however, is non-normalizable, and therefore does not correspond to any quantum mechanical systems with finite dimensional Hilbert space. Thus the conformal sector can only consist of fully degenerate states with $E = 0$, agreeing with the requirement imposed by diffeomorphism invariance. There is therefore no dynamics among such states.

SYK and tensor models are not conformal. At large $N$, they approximately have continuous density of state. Even so they only become approximately conformal at low energy. To make contact with the exactly conformal theory with $e^{S_0} \delta(E)$ density of state, we can think near $E = 0$ SYK and tensor models at large $N$ spreads the $e^{S_0}$ in a small interval of
size $E_{\text{gap}}$. The specific way of spreading is dictated by the density of states of the Schwarzian theory.

$$S = \frac{2\pi C}{J} \int d\tau \left( \frac{f'''(\tau)}{f'(\tau)} - \frac{3}{2} \frac{f''(\tau)^2}{f'(\tau)^2} \right).$$  \hspace{1cm} (1.2.2)

Theories with such conformal breaking terms are called nearly conformal theory. The SYK models and tensor models introduced below are the examples of theories with such a feature.

**Majorana SYK and tensor models**

The original SYK model is a quantum mechanical system that contains $N$ Majorana fermions $\psi^j$ and a all to all random interactions that couple $q$ fermions at a time:

$$H = \frac{j^{q/2}}{q!} \sum_{i_1i_2...i_q} \psi^{i_1} \psi^{i_2} ... \psi^{i_q}, \quad \{\psi^i, \psi^j\} = \delta^{ij}. \hspace{1cm} (1.2.3)$$

where the repeated indices are summed over, and these fermions obey the usual anti-commutator relations. The random couplings are drawn from a fixed Gaussian ensemble, with mean zero and variance

$$\langle J^2_{i_1i_2...i_q} \rangle = \frac{(q-1)!J}{N^{q-1}}. \hspace{1cm} (1.2.4)$$

At large $N$ the partition function becomes self-averaging. Equivalently by an ensemble averaging, we obtain the large $N$ effective action

$$\langle Z \rangle = \int D\psi \exp \left( \int dt \psi^i \partial_t \psi^i - \int dt dt' \psi^i(t) \psi^i(t') \langle J_{i_1...i_q}J_{i'_1...i'_q} \rangle \psi^{i_1}(t) \psi^{i_2}(t') \right)$$

$$= \int D\psi \exp \left( \int dt \psi^i \partial_t \psi^i - \frac{J^2}{qN^{q-1}} \int dt dt' \langle \psi^i(t) \psi^i(t') \rangle^q \right), \hspace{1cm} (1.2.5)$$

where $\langle J_{i_1i_2...i_q}J_{i'_1i'_2...i'_q} \rangle$ is the covariance matrix of the random variable. We note that the model has an emergent $O(N)$ global symmetry after ensemble averaging. Considering the
total derivative inside the path integral

\[
\int D\psi \frac{\delta}{\delta \psi^i(t_2)} \left( \psi^i(t_1) \exp \left(-S[\psi]\right) \right) = 0, \quad (1.2.6)
\]

we obtain the Schwinger Dyson equations for the fermions

\[
\langle -\frac{1}{N} \psi^i(t_1) \partial_{t_2} \psi^i(t_2) + \int dt_3 J^2 \frac{1}{N} \psi^i(t_1) \psi^j(t_3) \rangle \left( \frac{1}{N} \psi^j(t_2) \psi^j(t_3) \right)^{q-1} \right) = \delta(t_1 - t_2). \quad (1.2.7)
\]

Alternatively, the Schwinger Dyson equation can be obtained through bilocal variables. The action can be recast by inserting the identity

\[
1 = \exp \left( \int dt dt' \Sigma(t, t') \left( G(t, t') - \frac{1}{N} \psi^i(t) \psi^j(t') \right) \right), \quad (1.2.8)
\]

where \( \Sigma \) is the Lagrange multiplier that enforces the identification. Integrating out the fermions, we obtain the bilocal action in terms of the \( G, \Sigma \) variables.

\[
S_{\text{eff}}/N = \frac{1}{2} \log \det (\partial_t - \Sigma) - \int dt_1 dt_2 \left( \Sigma(t_1, t_2) G(t_1, t_2) - \frac{J^2}{q} G(t_1, t_2)^q \right). \quad (1.2.9)
\]

The equations of motion for \( G \) and \( \Sigma \) are given by

\[
\Sigma(t_1, t_2) = J^2 G(t_1, t_2)^q \quad \text{for} \quad \Sigma(t_1, t_2) = J^2 G(t_1, t_2)^q \quad (1.2.10)
\]

\[
\partial_t G(t_1, t_2) + \int dt_3 \Sigma(t_1, t_3) G(t_2, t_3) = \delta(t_1 - t_2). \quad (1.2.11)
\]

Note \( G \) satisfies the boundary condition provided by the anticommutating relation of the Majorana fermions:

\[
G(0^+) = \frac{1}{2}, \quad G(0^-) = -\frac{1}{2}. \quad (1.2.12)
\]

We notice that if the UV kinetic term \( \partial_t G \) in (1.2.11) can be ignored, the equations (1.2.10-1.2.11) are in fact invariant under the transformation induced by arbitrary reparametriza-
tions $t \to f(t)$:

$$G(t_1, t_2) \to f'(t_1) \Delta f'(t_2) \Delta G(f(t_1), f(t_2)),$$

(1.2.13)

as long as $\Delta = \frac{1}{q}$, and $\Delta$ can be thought of the conformal dimension of the elementary fermions.

Note such a reparametrization symmetry is only approximate in the infrared, because we ignore the effect of the UV kinetic term. Of course, the reparametrization symmetry is spontaneously broken by any solutions $G_c(t)$ that we pick. However, once adding back the UV kinetic term, the reparametrization symmetry gets explicitly broken. Among all approximate solutions, it singles out a particular solution that agrees with the exact solution at long distance.

A particularly interesting solution in the infrared is given by a power law Ansatz,

$$G_c(t_1, t_2) = \frac{b \text{sgn}(t_1 - t_2)}{|t_1 - t_2|^{2\Delta}},$$

(1.2.14)

where the value of two point coefficient is determined by the Schwinger Dyson equations to be

$$J^2 b^8 = \frac{1}{\pi} \left( \frac{1}{2} - \frac{1}{q} \right) \tan \left( \frac{\pi}{q} \right).$$

(1.2.15)

In a conformal field theory the coefficients of two point correlators can always be normalized by an overall scale transformation. In contrast, the coefficient in two point correlator (1.2.14) can not be rescaled, as the conformal symmetry is only approximate at long distance. A re-scaling would be incompatible with the UV boundary condition (1.2.12). We will see a case in chapter 4 where the coefficients are not determined by the infrared equations of motions, but rather determined by the full Schwinger Dyson equations.

In fact, we can verify that the solution (1.2.14) agrees with the full solution at long distance by numerically solving the Schwinger Dyson equations.

An important aspect of conformal field theories is the operator spectrum. It is a crucial part of the conformal data. From the nearly conformal solution (1.2.14) and Schwinger
Dyson equations (1.2.10-1.2.11), we may obtain the spectrum of operators bilinear in the fermions, of the form

\[ O_n = \psi^i \partial_t^{2n+1} \bar{\psi}^i, \quad n \in \mathbb{Z}_{\geq 0}, \quad (1.2.16) \]

where even number of derivatives always provide a descendent. To obtain their dimensions, we note that the Dyson-Schwinger equation (1.2.7) is an operator equation, and we may insert additional operators at the cost of additional contact terms. However, an operator inserted at infinite does not produce additional contact terms, and thus for any operator \( O \), the three point function

\[ \langle \frac{1}{N} \psi^i(t_1) \bar{\psi}^i(t_2) O(\infty) \rangle = \frac{c \text{ sgn}(t_1 - t_2)}{|t_1 - t_2|^{2\Delta - h}} \quad (1.2.17) \]

solves the Schwinger Dyson equation, where the functional form on the right hand side is constrained by the conformal invariance, and \( h \) is the conformal dimension of \( O \). In order for the three point function to be non-vanishing, \( O \) must be bilinear in the fermions. The constraint imposed by the Schwinger Dyson equations would provide a necessary condition for the dimensions of the bilinear operators.

Up to now we do not assume any symmetry besides translation in \( G, \Sigma \) variables. Since \( \psi^i \)'s are Majorana fermions, the correlator is always anti-symmetric upon time reflections,

\[ G(t) = -G(-t). \quad (1.2.18) \]

As we will see, it’s an important restriction as when there is no such restrictions, the system obeys the same equations as those of the bipartite model, which is discussed in Chapter 2 as a special case. In fact, there is an instability along the symmetric direction. Now we assume the correlators satisfy the anti-symmetric constraint, and consider the equations of motion in the infrared, where we assume that we could neglect the UV kinetic term in (1.2.11).

Now we work out the bilinear spectrum, considering variation of \( \delta G \) over \( G_c \), with the
form (1.2.17), and imposing that it satisfies the infrared Schwinger Dyson equation. To simplify notation, we shall denote the convolution as $\star$, and the variation is given by

$$\delta(\Sigma \star G) = \delta\Sigma \star G + \Sigma \star \delta G = 0.$$  

(1.2.19)

Convolute against $G$ from the left and we obtain

$$\delta G = J^2(q - 1)G \star (G^{q-2}\delta G) \star G,$$  

(1.2.20)

and substitute (1.2.14, 1.2.17) into (1.2.20) we obtain the eigenvalue

$$k_c(h) = J^2b^2(q - 1) \int dt_3 dt_4 \frac{\text{sgn}(t_{13})\text{sgn}(t_{42})\text{sgn}(t_{34})}{|t_{13}|^{2\Delta}|t_{24}|^{2\Delta}|t_{34}|^{2(q-1)\Delta-h}}.$$  

(1.2.21)

The integral can be most conveniently evaluated in the momentum space as convolutions become simple products. The constraint on conformal dimension $h$ is found to be

$$k_c(h) = \frac{\pi(q - 1)(q - 2)\Gamma\left(\frac{2}{q} - h\right)\sin\left(\frac{\pi}{q}(h - \frac{2}{q})\right)}{q \sin\left(\frac{\pi h}{2} + \frac{\pi}{q}\right)\sin\left(\frac{2\pi}{q}\right)\Gamma\left(2 - h - \frac{2}{q}\right)\Gamma\left(\frac{2}{q}\right)^2}, \quad k_c(h) = 1.$$  

(1.2.22)

We note that $h = 2$ is always a solution for arbitrary $q$. Such corresponds to operator $O^0$ in (1.2.16), which is the Hamiltonian. This is the mode responsible for reparametrizations.

Other operators obtain a small correction compared to the sum of the conformal dimensions of the fermion in the infrared and that of the derivatives. When $n$ is large the dimension approaches to $2\Delta + 2n + 1$.

So far we have been focusing on the conformal sectors of the SYK model. As we have discussed, the emergent reparametrization symmetry is spontaneously broken by $G_c(t)$, and explicitly broken at shorter distance. The effective action that describes the explicit breaking of reparametrization symmetry is known as the Schwarzian. It is convenient to describe the
theory at finite temperature, using the map \( f(t) = \tan \frac{\phi(t)}{2} \),

\[
S_{\text{eff}} = \frac{2\pi N \alpha_s}{\beta J} \int_0^{2\pi} dt \left( -\text{Sch} \left( \tan \frac{\phi}{2}, \tau \right) \right), \quad \text{Sch}(f(t), t) = \frac{f''(t)}{f'(t)} - \frac{3}{2} \frac{f''(t)^2}{f'(t)^2}. \tag{1.2.23}
\]

and such term is the leading local term that is invariant under \( SL(2, \mathbb{R}) \) transformation, and describes the breaking of the reparametrization symmetry. At low temperature, free energy thus admits an expansion

\[
-\beta F/N = -\beta E_0 + S_0 + \frac{c}{2\beta} + \ldots, \quad c = \frac{4\pi^2 \alpha_s N}{J}, \tag{1.2.24}
\]

where \( E_0 \) is the ground state energy, and \( S_0 \) the zero temperature entropy.

The \( \frac{1}{\beta} \) term is due to the Schwarzian, and is the leading term that describes the thermodynamics at low temperature. In fact, this is exactly the same theory that describes the low temperature thermodynamics of a Schwarzschild black hole.

Note the terms sub-leading in \( \frac{1}{\beta} \) are in general non-analytic power of \( \beta \), due to bilocal actions of operators, whose contribution is given by

\[
S \sim \int d\tau_1 d\tau_2 \left( \frac{f'((\tau_1) f'((\tau_2))}{(f(\tau_1) - f(\tau_2))^2} \right)^h \sim \frac{1}{\beta^{2h-2}}, \tag{1.2.25}
\]

where \( h \) is determined by (1.2.22).

An important feature of the Schwarzian theory is that its partition function can be exactly computed at all \( \beta \) through a technique called fermionic localization. The partition function can be shown to be one-loop exact [22].

Since the model (1.2.3) involves random couplings, it is unconventional compared to higher dimensional quantum field theories. As an alternative model, a non-random model was first proposed by Witten [16] and a simpler version is discovered by Klebanov and Tarnopolsky [15]. They encapsulate the same nearly conformal features as the SYK models. Along the way, a novel large \( N \) limit was found, where a class of diagrams called melonic
diagrams dominate at large $N$. We will give a short review of the Klebanov-Tarnopolsky model and contrast it to the SYK model.

Figure 1.2: Left: The illustration of the tetrahedron interaction where each vertex represents a fermion and the tetrahedron shows how various indices contract. Right: The tetrahedron vertex for Feynman diagrams in thick lines.

The tensor model contains $N^3$ Majorana fermions $\psi^{abc}$ transforming in the fundamental representations of three distinct $O(N)$ groups, where $a, b, c$ are respectively $O(N)$ indices. The Hamiltonian is given by

$$H = \frac{g}{4} \bar{\psi}^{a_1 b_1 c_1} \psi^{a_1 b_2 c_2} \bar{\psi}^{a_2 b_1 c_2} \psi^{1_2 b_2 c_1}.$$  \hfill (1.2.26)

Such an interaction is usually referred as a tetrahedron interaction, due to the index structure (see figure 1.2). In such a model, we would like to restrict our attention to $O(N)^3$ singlet states. The global symmetry $O(N)^3$ should be compared with the emergent $O(N)$ symmetry in SYK model. In the case of tensor model the symmetry is exact and can be gauged.

Figure 1.3: The basic melonic correction to the propagators in resolved representation and in fat (triple line) representations.

Diagrammatic analysis shows that there exists a large $N$ limit where $\lambda^2 = g^2 N^3$ is fixed as $N \to \infty$, and the leading diagrams consist of ones called melonic diagrams. These diagrams can be iteratively defined to be any diagrams that can be produced by iterating over the basic melon shown in figure 1.4 on top of its propagators. It is proven that such graphs are
maximally in the large $N$ limit \cite{15}, and therefore at large $N$, it is enough to consider only melonic graphs. That greatly simplifies the theory and makes it solvable at large $N$.

Using melonic dominance, we can write down the Schwinger Dyson equation as an recursive equation when we sum over the melonic diagrams:

\[
G(t_1 - t_2) = G_0(t_1 - t_2) + \int dt_3 dt_4 g^2 N^3 G_0(t_1 - t_3) G(t_3 - t_4)^3 G(t_4 - t_2), \tag{1.2.27}
\]

where $G_0(t)$ is the bare propagator and $G(t)$ is the exact propagator. The Schwinger Dyson equation can be graphically represented as in figure 1.5.

\[
\begin{align*}
\tau_1 &\rightarrow \tau_2 = \tau_1 - \tau_2 + \tau_1 \quad \tau_2
\end{align*}
\]

Figure 1.5: A graphical representation of the Schwinger Dyson equation.

Since $G_0(t) = \frac{1}{2}\text{sgn}(t)$ and $\partial G_0(t) = 2\delta(t)$, we may take derivative against $t_1$ to obtain

\[
\partial_1 G(t) - \int dt_3 dt_4 G(t_1 - t_4)^3 G(t_4 - t_2) = \delta(t_1 - t_2), \tag{1.2.28}
\]

which is exactly equivalent to the Schwinger Dyson equation (1.2.11). Since the majority of the nearly conformal features follow from the Schwinger Dyson equations, we expect tensor model to enjoy the same features.

The major difference comes from the fact tensor model is not a random theory. When
consider quantities such as the spectral form factor

$$\langle Z(\beta + iT)Z(\beta - iT) \rangle$$  \hspace{1cm} (1.2.29)$$

the SYK model after ensemble averaging would produce a smooth curve, whereas tensor model due to the absence of ensemble averaging, would produce a more erratic curve similar to the curve produced by a single realization in SYK spectral form factor.

To be more parallel to our discussion of SYK models, we give an alternative derivation of the large $N$ Schwinger Dyson equation in the tensor model. I thank Igor R.Klebanov, Vladimir Narovlansky, and Fedor K. Popov for collaboration over the materials presented below.

We consider the Majorana tensor model where we fix $\lambda^2 = g^2 N^3$.

$$S = \int dt \left( \frac{1}{2} \psi^{abc}_{\partial t} \psi^{abc} + \frac{\lambda}{4 N^3/2} \psi^{a_1b_1c_1} \psi^{a_1b_2c_2} \psi^{a_2b_1c_2} \psi^{a_2b_2c_1} \right).$$  \hspace{1cm} (1.2.30)$$

We can insert into the path integral an identity

$$1 = \int D\Sigma DG \exp \left( \int dt \int dt' \Sigma(t, t') \left( G(t, t') - \frac{1}{N^3} \psi^{abc}(t) \psi^{abc}(t') \right) \right) + f(G(t, t') - f\left( \frac{1}{N^3} \psi^{abc}(t) \psi^{abc}(t') \right),$$  \hspace{1cm} (1.2.31)$$

where $f(G)$ is an arbitrary function by our choice. Such a function has to be well chosen in order for us to obtain a non-trivial $G\Sigma$ effective action at large $N$. We choose $f(G) = \frac{\lambda^2}{4} G^4$ in the following. We can start by introducing the partition function as

$$Z = \int D\Sigma \Phi(\Sigma) \Psi(\Sigma),$$  \hspace{1cm} (1.2.32)$$
where we divide the path integral into two parts: the fermionic integral

\[
\Phi(\Sigma) = \int D\psi^{abc} \exp \left[ -\frac{1}{2} \int dt \psi^{abc} \partial_t \psi^{abc} - \frac{\lambda}{4N^{3/2}} \int dt \psi^{a_1b_1c_1} \psi^{a_1b_2c_2} \psi^{a_2b_1c_2} \psi^{a_2b_2c_1} 
- \int dt dt' \frac{\lambda^2}{4N^9} (\psi^{abc}(t) \psi^{abc}(t'))^4 \right]
\]

(1.2.33)

and the integral over \( G \):

\[
\Psi(\Sigma) = \int DGD\Sigma \exp \left( \int d\tau d\tau' \left( -\frac{N^3}{2} \Sigma G + \frac{\lambda^2 N^3}{4} G(\tau,\tau')^4 \right) \right)
\]

(1.2.34)

Consider the fermionic integral. It has two interactions. The first interaction is the tetrahedral interaction which gives melonic structure, where every melon (before the dressings of the internal propagators) is given a factor of \( \lambda^2 \). In the second (bilocal in time) interaction, because of the \( N \) coefficient, it will contribute at leading order in \( N \) only when three out of the four pairs of fermions are eventually contracted to each other.

Each such insertion is exactly equivalent to a melon, with coefficient \( (-\lambda^2) \). So the two interactions cancel in any order in perturbation theory, because replacing one melon by this bi-local interaction results in the same diagram, but with a minus sign. The sum over the diagrams, except for zeroth order in perturbation theory, then vanishes.

This means that the fermionic integral \( \Phi(\Sigma) \) at large \( N \) can be approximated by

\[
\Phi(\Sigma) = \exp \left( \frac{N^3}{2} \log \det (\partial_\tau - \Sigma(\tau, \tau')) \right),
\]

(1.2.35)

followed by non-melonic corrections subleading in \( N \). Therefore, at leading order in \( N \) we get the \( G\Sigma \) action as in SYK:

\[
Z = \int DGD\Sigma \exp \left( \frac{N^3}{2} \log \det (\partial_\tau - \Sigma(\tau, \tau')) - \frac{N^3}{2} \int d\tau d\tau' \left( \Sigma G - \frac{\lambda^2}{2} G(\tau, \tau')^4 + O(1/N) \right) \right).
\]

(1.2.36)
It is worth noting that although the tensor models and SYK models share the same features at large $N$, they receive distinct $\frac{1}{N}$ corrections. See more discussion in [38].

1.3 Overview

Besides the Schwarzian sector, the SYK and tensor models also exhibit rich physics in their conformal sector. In particular, the conformal sectors sometimes exhibit instabilities. In this thesis we encounter two type of instabilities observed in nearly conformal theories, whose fates would be determined in the following chapters of this work.

In the first case an instability is found in the bilinear spectrum where the scaling dimension of some bilinear operator becomes complex. This means the nearly conformal theory is not unitary, as unitarity requires the scaling dimension to be real and non-negative. For example such a complex mode is observed in the bipartite model [12]. In Chapter 2, we consider a more general version of the bipartite model, and we show that the operator with complex scaling dimension in fact condenses and acquires a VEV. Such an VEV spontaneously breaks a $\mathbb{Z}_2$ symmetry. Chapter 2 is based on work with Jaewon Kim, Igor R. Klebanov and Grigory Tarnopolsky [1]. In Chapter 3, we extend the analysis to a model with $U(1) \times U(1)$ symmetry. By tuning the coupling one of the operator charged under the axial $U(1)$ acquires a complex scaling dimension. The operator condenses in the low temperature, spontaneously breaking the $U(1)$ symmetry. Chapter 3 is based on work with Igor R. Klebanov, Alexey Milekhin, and Grigory Tarnopolsky [2]. In both cases, the phase transition is of second order, and the symmetry broken phase carries a large entropy.

The second type of instability is observed at non zero charge in complex SYK [39, 40], where there is no obvious problem in the bilinear spectrum. However the solution tunnels to a low entropy phase. Such transitions are of first order, characterized by a high entropy to low entropy transition.

In Chapter 4, we study $\mathcal{N} = 2$ supersymmetric SYK models with complex fermions, at
non-zero charge. We also propose a new $\mathcal{N} = 2$ supersymmetric SYK model with multiple $U(1)$ symmetries. In both models, a conformal solution with a Schwarzian mode emerges at low temperatures. However, in both cases the fermion scaling dimension depends on the background charges. For a critical charge we find a high-to-low entropy phase transition. As opposed to complex SYK, in $\mathcal{N} = 2$ SYK this transition has a simple interpretation: the fundamental fermion scaling dimension violates the unitarity bound, and the conformal solution becomes invalid or disappears. Chapter 4 is based on work with Matthew Heydeman and Gustavo Joaquin Turiaci.

Some of the results of the thesis are presented at Caltech and at the conference Strings 2019 in Brussels.
Chapter 2

The spontaneous breaking of $\mathbb{Z}_2$ symmetry

As introduced in Chapter 1.2, SYK and tensor models are solvable at large $N$ via the melonic Schwinger-Dyson equations [14, 26, 41–43], and has a nearly conformal phase at low energy. They shed new light on the dynamics of two-dimensional black holes [21, 44–46].

These models may also have applications to a range of problems in condensed matter physics, including the strange metals [25, 47–51]. With such applications in mind, it is interesting to study various dynamical phenomena in the SYK and tensor models. For example, phase transitions in such models have been studied in [40, 52, 53].

In this chapter we identify a simple setting where spontaneous symmetry breaking can occur: two SYK or tensor models coupled via a quartic interaction. We take this interaction to be purely melonic (i.e. tetrahedral in the tensor model case), so that the symmetry breaking can be deduced from the large $N$ Schwinger-Dyson equations. In this case, the symmetry breaking is triggered by a complex mode in the bilinear spectrum of the coupled system. In the infrared, the operator with complex scaling dimension condenses, and acquires a vacuum expectation value that spontaneously breaks the $\mathbb{Z}_2$ symmetry.
In the random case, we will study two coupled SYK models with the Hamiltonian

\[ H = \frac{1}{4!} J_{ijkl} (\chi^i_1 \chi^j_1 \chi^k_1 \chi^l_1 + \chi^i_2 \chi^j_2 \chi^k_1 \chi^l_2 + 6\alpha \chi^i_1 \chi^j_1 \chi^k_2 \chi^l_2) \]  

(2.0.1)

where the Majorana fermions are \( \chi^i_1 \) and \( \chi^i_2 \) with \( i = 1, \ldots, N_{\text{SYK}} \), and \( J_{ijkl} \) is a fully anti-symmetric real tensor with a Gaussian distribution. We will show that the real parameter \( \alpha \) may be restricted to the range \(-1 \leq \alpha \leq 1/3\) by a duality symmetry. This quartic Hamiltonian, which couples \( 2N_{\text{SYK}} \) Majorana fermions, is invariant under an anti-unitary particle-hole symmetry \([54–59]\) generated by \( \mathcal{P} \); see eq. (2.2.10). However, we will show that for \(-1 \leq \alpha < 0\) this \( \mathbb{Z}_2 \) symmetry is spontaneously broken when \( N_{\text{SYK}} \) is divisible by 4 and taken to infinity.\(^b\) In this limit the fermion number operator \( Q = i\chi^i_1 \chi^i_2 \) acquires an expectation value. This leads to a gapped phase in two coupled SYK models similar to that found by Maldacena and Qi \([60]\); however, instead of the quartic they assumed a quadratic coupling term \( \mu \mathcal{O} \) which breaks the \( \mathbb{Z}_2 \) symmetry explicitly. This gapped phase was argued to be dual to a traversable wormhole in two-dimensional gravity \([61, 62]\), and our model (2.0.1) may have a similar interpretation for \( \alpha < 0 \).

As we show in section 2.1.4, a sign of instability of the conformal phase for \(-1 \leq \alpha < 0\) is the presence of a complex scaling dimensions of the form \( 1/2 + if(\alpha) \). Appearance of complex dimensions with real part equal to \( d/2 \) for some single-trace operators is a common phenomenon in large \( N \) models \([63–67]\). Via the AdS/CFT correspondence \([8–10]\), such operators are related to scalar fields which violate the Breitenlohner-Freedman stability bound \([68]\). The complex scaling dimensions have been observed in bosonic tensor models \([69, 70]\), as well as in a complex fermionic model introduced in \([15]\) following the work in \([17]\). This fermionic model is often called “bipartite” because of the two types of interaction vertices (black and white) arranged in an alternating fashion, since the propagator must connect different vertices. The bipartite model was further studied in \([12]\) and shown to possess a

\(^a\)As usual, all repeated indices are summed over.

\(^b\)When \( N_{\text{SYK}} \) is finite and not divisible by 4, so that the total number of Majorana fermions is not divisible by 8, the particle-hole symmetry is broken by a discrete anomaly \([54–59]\).
complex scaling dimension of the operator $\bar{\psi}^{abc} \psi^{abc}$. Here we generalize this tensor model to one with a continuous parameter $\alpha$ in such a way that the bipartite model corresponds to $\alpha = -1$. This $O(N)^3$ symmetric model for Majorana fermions $\psi_1^{abc}$ and $\psi_2^{abc}$, with $a, b, c = 1, \ldots, N$, has Hamiltonian

$$H = \frac{g}{4} \left( \psi_1^{a_1b_1c_1} \psi_1^{a_1b_2c_2} \psi_1^{a_2b_1c_2} + \psi_2^{a_1b_1c_1} \psi_2^{a_1b_2c_2} \psi_2^{a_2b_1c_2} \right)$$

$$+ \frac{g\alpha}{2} \left( \psi_1^{a_1b_1c_1} \psi_1^{a_1b_2c_2} \psi_2^{a_2b_1c_2} + \psi_1^{a_1b_1c_1} \psi_1^{a_1b_2c_2} \psi_2^{a_2b_1c_2} \psi_2^{a_2b_2c_1} \right) .$$

For $\alpha = 0$ this describes two decoupled copies of the basic Majorana $O(N)^3$ model with the tetrahedral interaction [15]. The coupling term proportional to $\alpha$ preserves its discrete symmetries and also has the tetrahedral structure, i.e. every two tensors have only one index contraction, so that the model (2.0.2) is melonic. It is the tensor counterpart of the coupled SYK model (2.0.1), and in the large $N$ limit it is governed by the same Schwinger-Dyson equations for the two-point and four-point functions.

In section 2 we derive the Schwinger-Dyson equations and use them to study the scaling dimensions of various $O(N)^3$ invariant fermion bilinears. We also exhibit a duality symmetry which allows us to restrict the model to the range $-1 \leq \alpha \leq 1/3$. The nearly conformal phase of the theory is stable for $0 \leq \alpha \leq 1/3$, but it is unstable for $-1 \leq \alpha < 0$ as signaled by the complex scaling dimension of operator $i\bar{\psi}_1^{abc} \psi_2^{abc}$. The true behavior of the theory with negative $\alpha$ is the spontaneous breaking of the particle-hole $\mathbb{Z}_2$ symmetry. In section 3 we demonstrate this symmetry breaking using mostly the coupled SYK model (2.0.1). In section 2.2.1 and 2.2.2 we numerically study the large $N$ Schwinger-Dyson equations and exhibit the exponential decay of correlators at low temperature. We also ascertain the existence of second-order phase transitions by numerically computing the free energy. In section 2.2.3 we study the numerical spectrum at finite $N_{\text{SYK}}$ and show that for $-1 \leq \alpha < 0$ there is a gap separating two nearby lowest states from the rest of the spectrum. For $N_{\text{SYK}}$ divisible by 4 there is also a gap between the two lowest states, consistent with the general results on
Majorana systems [54–59], but this gap vanishes in the large $N_{\text{SYK}}$ limit. Then these two lowest states should become degenerate and give rise to the two inequivalent vacua, which are present due to the spontaneous breaking of the particle-hole symmetry.

This means that the low-temperature entropy is large for $\alpha > 0$ but vanishes for $\alpha < 0$. It is tempting to suggest that the $\alpha < 0$ theory is dual to a wormhole. This sensitivity to the sign of the interaction coupling two CFTs is like in [61], where the traversable wormhole appears only for one of the signs.\(^c\)

### 2.1 Schwinger-Dyson Equations and Scaling Dimensions

In this section we study the two-flavor tensor model with Hamiltonian (2.0.2).\(^d\) It can be compactly written in the form

$$H = \frac{1}{4!} J_{IJKL} \left( \psi_I^I \psi_J^J \psi_K^I + \psi_I^I \psi_J^J \psi_K^I + 6 \alpha \psi_I^I \psi_J^I \psi_K^J \psi_L^L \right),$$  

(2.1.1)

where the capital letters are a shorthand notation for three tensor indices: $I = a_1 b_1 c_1$, $J = a_2 b_2 c_2$, etc, and the non-random tetrahedral tensor coupling consists of six terms

$$J_{IJKL} = g \sum_{\sigma \in S_3} \text{sgn}(\sigma) \delta_{a_1 a_2(2)} \delta_{b_1 b_2(3)} \delta_{c_1 c_2(4)} \delta_{a_3 a_4(2)} \delta_{b_3 b_4(4)} \delta_{c_3 c_4(2)} \delta_{a_5 a_6(3)} \delta_{a_7 a_8(4)}. $$  

(2.1.2)

The tensor $J_{IJKL}$ is antisymmetric under permutation of indices $I, J, K, L$ and has a tetrahedron topology as shown in figure 2.1. In the form (2.1.1) the tensor model Hamiltonian is transparently similar to the SYK one (2.0.1). In terms of the complex tensors

$$\psi^I = \frac{1}{\sqrt{2}} (\psi_1^I + i \psi_2^I), \quad \overline{\psi}^I = \frac{1}{\sqrt{2}} (\psi_1^I - i \psi_2^I)$$  

(2.1.3)

\(^c\)On the other hand, in the approach of [60], where the quadratic term $\mu Q$ was added to couple the two SYK models, the gap (and therefore the wormhole) appeared for either sign of $\mu$.

\(^d\)This section is based in part on J.K.’s Princeton University senior thesis [71].
the Hamiltonian assumes the form

\[ H = \frac{1}{4!} J_{IJKL} \left( \frac{1 - 3\alpha}{2} \left( \psi^I \psi^J \psi^K \psi^L + \bar{\psi}^I \bar{\psi}^J \bar{\psi}^K \bar{\psi}^L \right) + 3(1 + \alpha) \bar{\psi}^I \bar{\psi}^J \psi^K \psi^L \right) \]  

(2.1.4)

\[ J_{IJKL} = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{Figure 2.1: Pictorial representation of the antisymmetric tensor } J_{IJKL}.
\end{array}
\end{array}
\end{array}\]

The Hamiltonian (2.1.1) is invariant under the \( O(N)^3 \) transformation

\[ \psi_i^{abc} \rightarrow A^a_i B^b_j C^c_i \psi_i^{a'b'c'}, \]  

(2.1.5)

where \( A, B, \) and \( C \) are orthogonal matrices. In addition, it has a particle-hole symmetry\(^e\) \( Z_2 \) symmetry generated by [54–59],

\[ \mathcal{P} = K \prod_I (\psi^I + \bar{\psi}^I) \]  

(2.1.6)

where \( K \) is the anti-linear operator \( KiK = -i \) and we assume that \( \psi^I \) and \( \bar{\psi}^I \) are real with respect to \( K \). The fermion number operator

\[ Q = i\psi_1^I \psi_2^I = \frac{1}{2} [\bar{\psi}^I, \psi^I] \]  

(2.1.7)

does not in general commute with \( H \), but it is conserved mod 4. The particle-hole symmetry is not anomalous only if the total number of fermions \( 2N^3 \) is a multiple of 8, i.e. when \( N \) is even [54–59]. Even in this case, we will argue that in the large \( N \) limit the symmetry is spontaneously broken for \(-1 \leq \alpha < 0\) because \( Q \) acquires an expectation value.

\(^e\)The Hamiltonian also has discrete symmetries which do not involve \( K \), which combine into the dihedral group \( D_4 \). This is discussed in detail for the coupled SYK counterpart in section 2.2 and in the Appendix.
2.1.1 Duality in the Two-Flavor Models

In this section we show that the two-flavor models with different values of $\alpha$ can be equivalent. We will demonstrate this explicitly in the tensor model case (2.1.1), but the SYK case (2.0.1) works analogously. Let us perform the following transformation on the Majorana fermions:

$$\psi^I_1 = \frac{1}{\sqrt{2}} (\tilde{\psi}^I_1 + \tilde{\psi}^I_2), \quad \psi^I_2 = \frac{1}{\sqrt{2}} (\tilde{\psi}^I_1 - \tilde{\psi}^I_2).$$

(2.1.8)

It preserves the anticommutation relations, and turns the Hamiltonian (2.1.1) into

$$H = \frac{1}{4!} J_{IJKL} \frac{1 + 3\alpha}{2} \left( \tilde{\psi}^I_1 \tilde{\psi}^J_1 \tilde{\psi}^K_1 \tilde{\psi}^L_1 + \tilde{\psi}^I_2 \tilde{\psi}^J_2 \tilde{\psi}^K_2 \tilde{\psi}^L_2 + \frac{6(1 - \alpha)}{1 + 3\alpha} \tilde{\psi}^I_1 \tilde{\psi}^J_1 \tilde{\psi}^K_2 \tilde{\psi}^L_2 \right).$$

(2.1.9)

Thus the energy levels are symmetric under the duality transformation

$$J \rightarrow \frac{1 + 3\alpha}{2} J, \quad \alpha \rightarrow \frac{1 - \alpha}{1 + 3\alpha}.$$  

(2.1.10)

Defining

$$\tilde{\alpha} = \frac{1}{2} (1 + 3\alpha), \quad \tilde{J} = J \sqrt{|\tilde{\alpha}|},$$

(2.1.11)

we find that the duality transformation

$$\tilde{\alpha} \rightarrow 1/\tilde{\alpha}, \quad \tilde{J} \rightarrow \tilde{J}$$

(2.1.12)

acts on the rescaled Hamiltonian $\tilde{H} = H/\sqrt{|\tilde{\alpha}|}$:

$$\tilde{H} = \frac{1}{4!} \tilde{J}_{IJKL} \left( \tilde{\psi}^I_1 \tilde{\psi}^J_1 \tilde{\psi}^K_1 \tilde{\psi}^L_1 + \tilde{\psi}^I_2 \tilde{\psi}^J_2 \tilde{\psi}^K_2 \tilde{\psi}^L_2 + \left( -2 + \frac{4}{\tilde{\alpha}} \right) \tilde{\psi}^I_1 \tilde{\psi}^J_1 \tilde{\psi}^K_2 \tilde{\psi}^L_2 \right).$$

(2.1.13)

This means that the fundamental domain is $-1 \leq \tilde{\alpha} \leq 1$. Thus, we may restrict $\alpha$ to the

---

1 Using antisymmetry of the tensor $J_{IJKL}$ one can operate with Majorana fermions as commuting variables but keeping order of $I, J, K, L$ indices fixed.

8 For the original Hamiltonian (2.1.13 this transformation rescales the energy levels. Therefore, our results for dimensionful quantities, like energy levels and Green functions, will not respect the duality under (2.1.12).
domain

\[-1 \leq \alpha \leq \frac{1}{3}. \tag{2.1.14}\]

The values of $\alpha$ outside of this domain are related to it by the duality. For $\alpha = -1$ the transformation (2.1.10) maps the theory into itself, but with $H \to -H$. In fact, the case $\alpha = -1$ corresponds to the complex bipartite model [12, 17]:

\[ H_{\alpha=-1} = 2 \frac{1}{4!} J_{IJKL} (\psi^I \psi^J \psi^K \psi^L + \bar{\psi}^I \bar{\psi}^J \bar{\psi}^K \bar{\psi}^L), \tag{2.1.15} \]

where we introduced the complex tensor $\psi^I = \frac{1}{\sqrt{2}} (\psi^I_1 + i \psi^I_2)$.

The theory with $\alpha = 1/3$ is mapped into itself by (2.1.10). In this case the Hamiltonian is

\[ H_{\alpha=\frac{1}{3}} = 4 \frac{1}{4!} J_{IJKL} \bar{\psi}^I \bar{\psi}^J \psi^K \psi^L, \tag{2.1.16} \]

which has $O(N)^3 \times U(1)$ symmetry. In the three-index notation

\[ H_{\alpha=\frac{1}{3}} = g \frac{2}{3} \left( \bar{\psi}^{a_1b_1c_1} \psi^{a_1b_2c_2} \psi^{a_2b_1c_2} \psi^{a_2b_2c_1} - \bar{\psi}^{a_1b_1c_1} \bar{\psi}^{a_2b_1c_2} \psi^{a_1b_2c_2} \psi^{a_2b_2c_1} \right). \tag{2.1.17} \]

This is different from the $SU(N)^2 \times O(N) \times U(1)$ symmetric complex tensor model [15]; the latter involves taking only the first term in this Hamiltonian.

### 2.1.2 Feynman rules and two-point functions

At first we list the Feynman rules which follow from the Hamiltonian (2.0.2). In figures (2.2) and (2.3) we define propagators and interaction vertices for the given two-flavour tensor model. Since the interaction terms have a tetrahedral tensor structure the melonic Feynman diagrams dominate in the large $N$ limit. Let us define bare two-point functions

\[ G_{11}(\tau_{12}) = \frac{1}{N^3} \langle T \psi^I_1(\tau_1) \psi^I_1(\tau_2) \rangle_0, \quad G_{22}(\tau_{12}) = \frac{1}{N^3} \langle T \psi^I_2(\tau_1) \psi^I_2(\tau_2) \rangle_0, \tag{2.1.18} \]
\( \langle \psi^I_1(\tau) \psi^I_2(\tau) \rangle_0 = \delta^{IJ} \tau^1 \tau^2 \)
\( \langle \psi^I_2(\tau) \psi^I_2(\tau) \rangle_0 = \delta^{IJ} \tau^1 \tau^2 \)

Figure 2.2: Bare propagators for the Majorana tensor fields. Each thick black solid or dashed line carries three tensor indices \( a, b, c \).

\[
\begin{align*}
L & \quad I \quad J \quad L \quad I \quad J \quad L \quad I \quad J \quad = J_{IJKL} \quad L \quad I \quad J \quad L \quad I \quad J \quad = \alpha J_{IJKL}
\end{align*}
\]

Figure 2.3: Interaction vertices.

where the sum over indices \( I \) is assumed. The leading melonic correction to the full two-point function \( G_{11} \) is represented in figure 2.4. Using that

\[
\begin{align*}
J_{IJKL}J_{IJKL} &= g^2(6N^6 - 18N^4 + 12N^3) \quad (2.1.19)
\end{align*}
\]

we find

\[
G_{11}(\tau_{12}) = G_{11}(\tau_{12}) + g^2N^3 \int d\tau_3 d\tau_4 G_{11}(\tau_{13})(G_{11}(\tau_{34})^3 + 3\alpha^2G_{11}(\tau_{34})G_{22}(\tau_{34})^2)G_{11}(\tau_{42}) + \ldots . \quad (2.1.20)
\]

A similar expression can be derived for \( G_{22} \). Since there is a symmetry \( \psi_1 \leftrightarrow \psi_2 \) we can assume that \( G_{11} = G_{22} = \mathbf{G} \) and obtain a Schwinger-Dyson equation for the full two-point function (see figure 2.5)

\[
\mathbf{G}(\tau_{12}) = G(\tau_{12}) + (1 + 3\alpha^2)g^2N^3 \int d\tau_3 d\tau_4 G(\tau_{13})\mathbf{G}(\tau_{34})^3\mathbf{G}(\tau_{42}) , \quad (2.1.21)
\]
where \( G(\tau) = \frac{1}{2} \text{sgn}(\tau) \) is the bare propagator.

\[
\frac{\tau_1}{\tau_2} = \frac{\tau_1}{\tau_3} + \frac{\tau_1}{\tau_4} \quad \text{for} \quad \tau_1 \neq \tau_2, \tau_3, \tau_4.
\]

Figure 2.5: Schwinger-Dyson equation for the full two-point function \( G(\tau_{12}) \).

In writing this Schwinger-Dyson equation we implicitly made an important assumption that the two-point functions

\[
G_{12}(\tau_{12}) = \frac{1}{N^3} \langle T\psi_1(\tau_1)\psi_2(\tau_2) \rangle, \quad G_{21}(\tau_{12}) = \frac{1}{N^3} \langle T\psi_2(\tau_1)\psi_1(\tau_2) \rangle,
\]

are zero \( G_{12}(\tau) = G_{21}(\tau) = 0 \). This follows from the \( \mathbb{Z}_2 \) symmetry \( \psi_2 \rightarrow -\psi_2 \). As we will see below, the \( \mathbb{Z}_2 \) symmetry can be spontaneously broken for some range of parameter \( \alpha \) and dimensionless coupling \( \beta J \), where \( \beta = 1/T \) is the inverse temperature and \( J^2 = g^2 N^3 \) is effective coupling constant.

Let us first assume that \( \mathbb{Z}_2 \) symmetry is not broken and analyze the SD equation (2.1.21). At large coupling constant \( \beta J \) and intermediate distances \( 1/J \ll \tau \ll \beta \) the solution to this equation is given by

\[
G(\tau) = \left( \frac{1}{4\pi(1 + 3\alpha^2)} \right)^{\frac{1}{4}} \frac{\text{sgn}(\tau)}{|J\tau|^{1/2}}.
\]

### 2.1.3 Scaling dimensions of bilinear operators

We can use the large \( N \) Schwinger-Dyson equations for the three-point functions to deduce the scaling dimensions of four families of bilinear operators:

\[
O_{1}^{2n+1} = \psi_1 \partial_\tau^{2n+1} \psi_1 + \psi_2 \partial_\tau^{2n+1} \psi_2, \quad O_{2}^{2n+1} = \psi_1 \partial_\tau^{2n+1} \psi_1 - \psi_2 \partial_\tau^{2n+1} \psi_2, \\
O_{3}^{2n+1} = \psi_1 \partial_\tau^{2n+1} \psi_2 + \psi_2 \partial_\tau^{2n+1} \psi_1, \quad O_{4}^{2n} = \psi_1 \partial_\tau^{2n} \psi_2 - \psi_2 \partial_\tau^{2n} \psi_1.
\]
where \( n = 0, 1, 2, \ldots \), and the sum over tensor indices is assumed.\(^{\text{h}}\) Each of these operators is invariant under the \( O(N)^3 \) symmetry, but they are distinguished by their transformations to discrete symmetry.

We take some operator \( O(\tau) \) and consider two three-point functions of the form

\[
v_{11}(\tau_1, \tau_2, \tau_0) = \langle \psi^I_1(\tau_1)\psi^I_1(\tau_2)O(\tau_0) \rangle, \quad v_{22}(\tau_1, \tau_2, \tau_0) = \langle \psi^I_2(\tau_1)\psi^I_2(\tau_2)O(\tau_0) \rangle, \tag{2.1.25}\]

where we assume summation over the index \( I \). In the large \( N \) limit the functions (2.1.25) obey the melonic Bethe-Salpeter equations. They are schematically represented in figure 2.6.

In the conformal limit one can ignore the first diagram on the right and obtain

\[
\begin{pmatrix} v_{11} \\ v_{22} \end{pmatrix} = \begin{pmatrix} K_{11,11} & K_{11,22} \\ K_{22,11} & K_{22,22} \end{pmatrix} \ast \begin{pmatrix} v_{11} \\ v_{22} \end{pmatrix}, \tag{2.1.26}\]

where assuming that \( G_{11} = G_{22} = G \) we find

\[
K_{11,11} = K_{22,22} = \frac{1 + \alpha^2}{1 + 3\alpha^2} K_c, \quad K_{11,22} = K_{22,11} = \frac{2\alpha^2}{1 + 3\alpha^2} K_c, \tag{2.1.27}\]

and \( K_c \) is the kernel of the SYK model, defined in conformal limit as

\[
K_c(\tau_1, \tau_2; \tau_3, \tau_4) = -\frac{3}{4\pi |\tau_{13}|^{2\Delta} |\tau_{24}|^{2\Delta} |\tau_{34}|^{2-4\Delta}} \operatorname{sgn}(\tau_{13})\operatorname{sgn}(\tau_{24}), \quad \Delta = \frac{1}{4}. \tag{2.1.28}\]

\(^{\text{h}}\)In the coupled SYK model (2.0.1) the same expressions for bilinear operators are applicable after replacement of \( \psi^I_A \) by \( \chi^I_A \), with \( A = 1, 2 \).
An arbitrary conformal three-point function of the form (2.1.25) with an operator of scaling dimension $h$ has the form

$$v_h(\tau_1, \tau_2, \tau_0) = \frac{c \text{sgn}(\tau_{12})}{|\tau_{01}|^h |\tau_{02}|^h |\tau_{12}|^{2\Delta-h}},$$

(2.1.29)

and obviously must be antisymmetric under $\tau_1 \leftrightarrow \tau_2$. This three-point function is an eigenvector of the kernel $K_c$ with the eigenvalue $g(h)$:

$$g(h) \int d\tau_3 d\tau_4 K_c(\tau_1, \tau_2; \tau_3, \tau_4) v_h(\tau_3, \tau_4, \tau_0) = v_h(\tau_1, \tau_2, \tau_0).$$

(2.1.30)

To solve (2.1.26) one has to find eigenvalues of the matrix and equate them to unity. This gives an equation for possible scaling dimensions. It easy to see that this matrix acquires diagonal form in the basis of vectors $v_{11} + v_{22}$ and $v_{11} - v_{22}$ and we find two equations for the scaling dimensions

$$g_A(h) = 1, \quad \frac{1 - \alpha^2}{1 + 3\alpha^2} g_A(h) = 1, \quad g_A(h) = -\frac{3\tan\left(\frac{\pi}{2}(h - \frac{1}{2})\right)}{2h - 1/2}.$$  

(2.1.31)

We conclude that the scaling dimensions of the operator $O_1^{2n+1} = \psi_1 \partial_{\tau}^{2n+1} \psi_1 + \psi_2 \partial_{\tau}^{2n+1} \psi_2$ satisfy $g_A(h) = 1$ and are independent of $\alpha$. They are given by well-known series $h = 2.00, 3.77, 5.68, 7.63, 9.60, \ldots$ and it eventually converges to $2n + \frac{3}{2}$.

On the other hand, scaling dimensions of the operator $O_2^{2n+1} = \psi_1 \partial_{\tau}^{2n+1} \psi_1 - \psi_2 \partial_{\tau}^{2n+1} \psi_2$ are given by $\frac{1-\alpha^2}{1+3\alpha^2} g_A(h) = 1$ and dependent on $\alpha$.

Note that scaling dimensions of $O_1$ operators are analogous to the flavourless real fermion model [15]. A sanity check can be done here. Let’s take $\alpha = 0$. In this limit the spectrum of $O_2$ converges to the spectrum of $O_1$, and this is as expected: since the two flavours of the fermions are decoupled at $\alpha = 0$, we expect the theory to be identical to [15].

\footnote{To take the integrals one should use star-triangle identities twice [26].}
Now consider the last possible three-point function

\[ v_{12}(\tau_1, \tau_2, \tau_0) = \langle \psi_I^\dagger(\tau_1) \psi_I^\dagger(\tau_2) O(\tau_0) \rangle. \]  
(2.1.32)

The melonic Bethe-Salpeter equation for this three-point function is represented in figure 2.7 and in the conformal limit, neglecting the first diagram on the right we get

\[ v_{12}(\tau_1, \tau_2, \tau_0) = \int d\tau_3 d\tau_4 \frac{2}{1 + 3\alpha^2} (\alpha K_c(\tau_1, \tau_2; \tau_3, \tau_4) - \alpha^2 K_c(\tau_1, \tau_2; \tau_4, \tau_3)) v_{12}(\tau_3, \tau_4, \tau_0). \]  
(2.1.33)

In this case there are two general possibilities for conformal three-point function, namely anti-symmetric and symmetric cases

\[ v^A_h(\tau_1, \tau_2, \tau_0) = \frac{c \text{sgn}(\tau_{12})}{|\tau_{01}|^h |\tau_{02}|^h |\tau_{12}|^{2\Delta - h}}, \quad v^S_h(\tau_1, \tau_2, \tau_0) = \frac{c \text{sgn}(\tau_{01})\text{sgn}(\tau_{02})}{|\tau_{01}|^h |\tau_{02}|^h |\tau_{12}|^{2\Delta - h}}. \]  
(2.1.34)

Therefore we finally find equations which determine spectrum of antisymmetric and symmetric operators

\[ \frac{2(\alpha + \alpha^2)}{1 + 3\alpha^2} g_A(h) = 1, \quad \frac{6(\alpha - \alpha^2)}{1 + 3\alpha^2} g_S(h) = 1, \quad g_S(h) = -\frac{1}{2} \frac{\tan(\frac{\pi}{2}(h + \frac{1}{2}))}{h - 1/2}. \]  
(2.1.35)

The scaling dimensions of the operators \( O_3 \) satisfy \( \frac{2(\alpha + \alpha^2)}{1 + 3\alpha^2} g_A(h) = 1 \), and that of the operators \( O_4 \) satisfy \( \frac{6(\alpha - \alpha^2)}{1 + 3\alpha^2} g_S(h) = 1 \).

We can check this result by comparing it to that of some other models. The complex bipartite fermion model has the Hamiltonian (2.1.15). It was found [12] that for the symmetric
sector

\[ g_{\text{sym}}(h) = \frac{3 \tan\left(\frac{\pi}{2} (h + \frac{1}{2})\right)}{2 (h - 1/2)} \] (2.1.36)

and indeed for \( \alpha = -1 \) we get \( \frac{6(\alpha - \alpha^2)}{1 + 3\alpha^2} g_S(h) = g_{\text{sym}}(h) \). To summarize, we have found that scaling dimensions of the operators (2.1.24) can be obtained by solving equations \( g_i(h) = 1 \), where

\[ (g_1(h), g_2(h), g_3(h), g_4(h)) = \left( g_A(h), \frac{1 - \alpha^2}{1 + 3\alpha^2} g_A(h), \frac{2\alpha(1 + \alpha)}{1 + 3\alpha^2} g_A(h), \frac{6\alpha(1 - \alpha)}{1 + 3\alpha^2} g_S(h) \right). \] (2.1.37)

The duality relation (2.1.10) is reflected in the behavior of functions \( g_i(h) \), which define scaling dimensions of the operators \( O_i \). Using (2.1.37) and (2.1.10) one finds

\[ (g_1(h), g_2(h), g_3(h), g_4(h)) \rightarrow (g_1(h), g_3(h), g_2(h), g_4(h)). \] (2.1.38)

Indeed, under \( \psi \rightarrow \tilde{\psi} \) the operators \( O_i \) transform as \( (O_1, O_2, O_3, O_4) \rightarrow (O_1, O_3, O_2, O_4) \).

### 2.1.4 Complex scaling dimensions

In this section, we examine if there exist any complex solutions of the equations \( g_i(h) = 1 \) defined in (2.1.37). If such a complex root exists, then a conformal primary has a complex scaling dimension, which leads to a destabilization of the model. Indeed, a complex scaling dimension of the form \( \Delta = \frac{1}{2} \pm if \) corresponds to a scalar fields in \( AdS_2 \) whose \( m^2 \) is below the Breitenlohner-Freedman bound \( m_{BF}^2 = -\frac{1}{4} \). Since \( \Delta = \frac{1}{2} \pm \sqrt{\frac{1}{4} + m^2} \) [8–10],

\[ m^2 = -\frac{1}{4} - f^2 < m_{BF}^2. \] (2.1.39)

In such a case one may expect “tachyon condensation” in AdS space. In the dual CFT the operator dual to the tachyon acquires an expectation value, leading to symmetry breaking. We will obtain some support to this picture.
First of all we notice that the functions $g_A(h)$ and $g_S(h)$ are real only if $h$ is real or $h = \frac{1}{2} + if$ for real $f$. Next it is easy to check that

$$-\frac{3\pi}{4} \leq g_A(1/2 + if) < 0, \quad -\infty \leq g_S(1/2 + if) < 0.$$  

(2.1.40)

Using the fact that $-\frac{1}{3} \leq \frac{1-\alpha^2}{1+3\alpha^2} \leq 1$ (and the same for $\frac{2\alpha(1+\alpha)}{1+3\alpha^2}$ due to duality) we conclude that equations $g_i(1/2 + if) = 1$ for $i = 1, 2, 3$ do not have solutions, thus scaling dimensions of the operators $O_1, O_2$ and $O_3$ are always real.

On the other hand, since $\frac{6\alpha(1-\alpha)}{1+3\alpha^2} < 0$ for negative $\alpha$, the equation $g_4(h) = 1$ has a solution $h = 1/2 + if(\alpha)$, where $f(\alpha)$ can be found from the equation

$$f \tanh(\pi f/2) = \frac{3\alpha(1-\alpha)}{1 + 3\alpha^2}.$$  

(2.1.41)

The plot of $f(\alpha)$ is shown in figure 2.8. For slightly negative $\alpha$ we find

$$f(\alpha) = \sqrt{-\frac{6\alpha}{\pi}} (1 + O(\alpha)),$$  

(2.1.42)

while $f(-1) \approx 1.5251$ in agreement with the result for the bipartite model found in [12].

![Figure 2.8: Imaginary part of the scaling dimension of the fermion number operator $Q$. At $\alpha = -1$ it reaches its maximum value $\approx 1.5251$.](image)

Thus, for $-1 \leq \alpha < 0$ there is one operator with a complex scaling dimension of the form $h = 1/2 + if(\alpha)$: the fermion number operator $Q = O_4^0$. This makes the conformal
large $N$ limit unstable. It is interesting to study the IR physics of the theory. As remarked at the beginning of the section, we may expect the operator whose scaling dimension is formally complex, to acquire a vacuum expectation value. Since this operator is odd under the symmetry (2.1.6) this symmetry is then spontaneously broken. In the next section we will study this symmetry breaking from the point of view of the generalized $G\Sigma$ action of the corresponding doubled SYK model.

2.2 Symmetry Breaking

In section 2.1.4, we showed that for the coupled tensor model (2.0.2) in the range $-1 \leq \alpha < 0$ the fermion number operator $Q = i\psi_1^J \psi_2^I$ has a complex scaling dimension, signaling an instability of the conformal phase of the model. In this section we show that this operator acquires a vacuum expectation value (VEV) in the true low-temperature phase of the large $N$ model. Based on this, it is tempting to make the following conjecture.

Conjecture. If the assumption of conformal invariance in a large $N$ theory leads to a single-trace operator with a complex scaling dimension of the form $d/2 + if$, then in the true low-temperature phase this operator acquires a VEV.

In our case, the $O(N)^3$ symmetry implies that

$$\langle i\psi_1^I \psi_2^J \rangle = \delta^{IJ} A, \quad (2.2.1)$$

where we used the short-handed notation $I = abc$, and $A$ is of order 1 in the large $N$ limit. This provides an explicit scale in the theory and leads to an exponential decay of correlation functions, which signify a gap in the spectrum. Furthermore, the VEV (2.2.1) implies that various discrete symmetries, including the particle-hole symmetry (2.1.6), the interchange symmetry between $\psi_1, \psi_2$, and the reflection symmetry $\psi_2 \rightarrow -\psi_2$, are spontaneously broken. Therefore, one should expect a second-order phase transition between the broken and
unbroken symmetry phases. In addition, the spontaneously broken symmetry also implies a ground state degeneracy in the large $N$ energy spectrum.\footnote{Due to a technicality we only expect a two-fold degeneracy although multiple $\mathbb{Z}_2$ have been broken. We will comment on this issue below.}

In this section we extensively analyze the phenomenon of symmetry breaking, sometimes using the SYK counterpart (2.0.1) of the $O(N)^3$ tensor model (2.0.2). The two models have many similarities at large $N$: they share the same Schwinger-Dyson equations, and the spectra of bilinear operators. The SYK formulation, however, is advantageous for the purpose of exact numerical diagonalizations: we can study cases where the integer $N_{\text{SYK}}$ is not the cube of an integer.

Let us first demonstrate the connection between the tensor model and the SYK counterpart. For the one-flavor $O(N)^3$ tensor model the analogous SYK model has the random tensor $J_{ijkl}$ which is fully antisymmetric. The mixed term $A_{ijkl}\chi_i^1\chi_j^1\chi_k^2\chi_l^2$ has only the symmetries

$$A_{ijkl} = -A_{jikl} = -A_{ijlk} = A_{klij},$$

(2.2.2)

which are the same as for the Riemann tensor. However, the full interaction term following from (2.0.2) is

$$A_{ijkl}(\chi_i^1\chi_j^1\chi_k^2\chi_l^2 + \chi_i^1\chi_j^1\chi_k^2\chi_l^2 + \chi_i^1\chi_j^1\chi_k^2\chi_l^2) = (A_{ijkl} + A_{iljk} + A_{iklj})\chi_i^1\chi_j^1\chi_k^2\chi_l^2. \quad (2.2.3)$$

Since $A_{ijkl} + A_{iljk} + A_{iklj}$ is fully antisymmetric due to (2.2.2), the mixed term has a fully antisymmetric random coupling. We will assume that it is proportional to the coupling in the diagonal term of (2.0.2), and are thus led to the random model (2.0.1). Introducing the complex combination $\psi^j = \frac{1}{\sqrt{2}}(\chi_1^j + i\chi_2^j)$, we may write the Hamiltonian as

$$H = \frac{1}{4!}J_{ijkl}\left(\frac{1 - 3\alpha}{2} \left(\psi^i\psi^j\psi^k\psi^l + \bar{\psi}^i\bar{\psi}^j\bar{\psi}^k\bar{\psi}^l\right) + 3(1 + \alpha)\bar{\psi}^i\bar{\psi}^j\psi^k\psi^l\right). \quad (2.2.4)$$

As usual, we will assume that each variable $J_{ijkl}$ has a gaussian distribution with variance
We will typically state energies in units of $J$, or equivalently set $J = 1$.

The duality symmetry described in section 2.1.1 applies to the coupled SYK model (2.0.1), and again allows us to restrict $\alpha$ to the range from $-1$ to $1/3$. There are two interesting limiting cases. For $\alpha = -1$ the transformation (2.1.10) maps $H \to -H$. This means that, for any random choice of $J_{ijkl}$ the energy spectrum is exactly symmetric under $E \to -E$. This can be seen in the histograms of the spectrum shown in fig. (2.18); in particular, there are many states whose energy is exactly zero. For $\alpha = -1$ the model is a random counterpart of the complex bipartite model:

$$H_{\alpha=-1} = \frac{2}{4!} J_{ijkl} \left( \psi^i \psi^j \psi^k \psi^l + \bar{\psi}^i \bar{\psi}^j \bar{\psi}^k \bar{\psi}^l \right). \quad (2.2.5)$$

The fermion number operator

$$Q = i \chi_1^j \chi_2^j = \frac{1}{2} [\bar{\psi}^j, \psi^j] \quad (2.2.6)$$

does not in general commute with $H$, but it is conserved mod 4. For $\alpha = 1/3$, however, we find the Hamiltonian

$$H_{\alpha=1/3} = \frac{4}{4!} J_{ijkl} \bar{\psi}^i \bar{\psi}^j \psi^k \psi^l, \quad [Q, H_{\alpha=1/3}] = 0. \quad (2.2.7)$$

Thus, we have enhanced $U(1)$ symmetry $\psi^j \to e^{i\gamma} \psi^j$. We note that, for $\alpha = 1/3$ the scaling dimension of operator $Q = O_4^0$ is $h = 1$ consistent with charge conservation. Also, here $g_2(h) = g_3(h)$, so that the scaling dimensions of $O_2^{2n+1}$ and $O_3^{2n+1}$ are equal. This is because

$$O_2^{2n+1} + i O_3^{2n+1} = 2 \psi^j \partial_t^{2n+1} \psi^j. \quad (2.2.8)$$

Furthermore, the transformation (2.1.10) maps $H_{\alpha=1/3}$ into itself, so the theory is selfdual.

---

$k$This model is similar to the complex SYK model [25], but in (2.2.7) the coupling $J_{ijkl}$ is taken to be fully antisymmetric.
For general \( \alpha \), the model (2.0.1) has multiple discrete symmetries, which are discussed in more detail in the appendix. These discrete symmetries can be spontaneously broken due to a VEV of \( Q \) if \( Q \) is not invariant under them. In the model (2.0.1), there are two symmetries that are not broken by a VEV of \( Q \): the anti-unitary time-reversal symmetry \( K \), and a \( \mathbb{Z}_4 \) symmetry generated by \( \frac{\pi}{2} \) rotation \( R \) in \( \chi_1, \chi_2 \).

\[
R\chi_1 R^\dagger = \chi_2, \quad R\chi_2 R^\dagger = -\chi_1 . \tag{2.2.9}
\]

They both preserve \( Q \). The model (2.0.1) also has multiple reflection symmetries that are spontaneously broken by the VEV of \( Q \), which we list in the appendix. In fact all unitary discrete symmetries of the model (2.0.1) form the Dihedral group of order 8, \( D_4 \). In our case, any two broken symmetries that can be related by an unbroken symmetry do not produce extra ground state degeneracy, and therefore it is enough to focus on one of them.

Let us focus on the particle-hole symmetry \([54–59]\) generated by

\[
\mathcal{P} = K \prod_{i=1}^{N_{\text{SYK}}} (\psi_i + \bar{\psi}_i) , \quad \mathcal{P}^2 = (-1)^{N_{\text{SYK}}(N_{\text{SYK}}-1)/2} . \tag{2.2.10}
\]

It acts on the fermion number as

\[
\mathcal{P}Q\mathcal{P} = -\mathcal{P}^2 Q . \tag{2.2.11}
\]

For \( N_{\text{SYK}} \) not divisible by 4, there is a two-fold degeneracy of the ground state in section 2.2.3, due to an anomaly in the particle-hole symmetry \([54–59]\). For \( N_{\text{SYK}} \) divisible by 4 this symmetry is not anomalous, and we find a non-degenerate ground state, which is followed by a nearby state when \( -1 \leq \alpha < 0 \). The two lowest states become degenerate in the large \( N_{\text{SYK}} \) limit, and they are separated by a gap from the remaining states. This leads to a spontaneous symmetry breaking through the formation of an expectation value of \( Q \). We will demonstrate this effect by solving the large \( N_{\text{SYK}} \) Schwinger-Dyson equations for the
Green functions, and with diagonalizations at finite $N_{\text{SYK}}$.

### 2.2.1 Schwinger-Dyson equations and the effective action

In this section we derive the large $N_{\text{SYK}}$ effective action of $G\Sigma$ type, and the Schwinger-Dyson equations, for the coupled SYK model (2.0.1). Following [60], we introduce bi-local variables

$$G_{ab}(\tau, \tau') = \frac{1}{N_{\text{SYK}}} \langle T \chi^i_a(\tau) \chi^j_b(\tau') \rangle ,$$  \hspace{1cm} (2.2.12)

and the corresponding Lagrange multipliers $\Sigma_{ab}(\tau, \tau')$, where $a, b = 1, 2$. The effective action is given by

$$-\frac{\beta S_{\text{eff}}}{N_{\text{SYK}}} = \log \text{Pf}(\partial_\tau \delta_{ab} - \Sigma_{ab}) - \frac{1}{2} \int d\tau d\tau' \left( \sum_{a,b} \Sigma_{ab}(\tau, \tau') G_{ab}(\tau, \tau') - \frac{J^2}{4} \left( \sum_{a,b} G_{ab}(\tau, \tau')^4 \right) \right.$$  

$$+ 6\alpha (G_{12}^2(\tau, \tau') + G_{21}^2(\tau, \tau'))(G_{11}^2(\tau, \tau') + G_{22}^2(\tau, \tau'))  + 6\alpha^2 (G_{11}^2(\tau, \tau')G_{22}^2(\tau, \tau')$$
$$+ G_{12}^2(\tau, \tau')G_{21}^2(\tau, \tau') + 4G_{11}(\tau, \tau')G_{22}(\tau, \tau')G_{12}(\tau, \tau')G_{21}(\tau, \tau')) \right) .$$

By translation invariance

$$G_{ab}(\tau, \tau') = G_{ab}(\tau - \tau') , \quad \Sigma_{ab}(\tau, \tau') = \Sigma_{ab}(\tau - \tau') .$$  \hspace{1cm} (2.2.13)

We also have the general properties

$$G_{11}(\tau) = -G_{11}(\tau) , \quad G_{22}(\tau) = -G_{22}(\tau) , \quad G_{12}(\tau) = -G_{21}(\tau) .$$  \hspace{1cm} (2.2.14)
The Schwinger Dyson (SD) equations for the two point functions assume the form

\[ \partial_\tau G_{11}(\tau) - \int d\tau' (\Sigma_{11}(\tau - \tau')G_{11}(\tau') + \Sigma_{12}(\tau - \tau')G_{21}(\tau')) = \delta(\tau) , \]
\[ \partial_\tau G_{12}(\tau) - \int d\tau' (\Sigma_{11}(\tau - \tau')G_{12}(\tau') + \Sigma_{12}(\tau - \tau')G_{22}(\tau')) = 0 , \]
\[ J^{-2} \Sigma_{11} = G_{11}^3 + 3\alpha G_{11}(G_{12}^2 + G_{21}^2) + 3\alpha^2 G_{11}G_{22}^2 + 6\alpha^2 G_{22}G_{12}G_{21} , \]
\[ J^{-2} \Sigma_{12} = G_{12}^3 + 3\alpha G_{12}(G_{11}^2 + G_{22}^2) + 3\alpha^2 G_{12}G_{21}^2 + 6\alpha^2 G_{11}G_{22}G_{21} , \]  
(2.2.15)

and similarly for \( 1 \leftrightarrow 2 \). These equations and the effective action are invariant under \( 1 \leftrightarrow 2 \) and \( G_{12} \rightarrow -G_{12}, G_{21} \rightarrow -G_{21} \).

2.2.2 Solutions of Schwinger-Dyson equations and symmetry breaking

For \( 0 \leq \alpha \leq 1/3 \) there are no operators with complex scaling dimensions, so it is consistent to assume that the discrete symmetries are unbroken and set \( G_{12} = 0 \), and \( G_{11} = G_{22} \), to obtain a nearly conformal solution in the low energy limit. However, the appearance of a complex scaling dimension for \( -1 \leq \alpha < 0 \) shows that such a conformal phase is unstable.

We will show that, in this range of \( \alpha \) the true phase of the theory exhibits spontaneous symmetry breaking. In order to exhibit it, we have to allow the possibility that \( G_{12}(\tau) \neq 0 \).

The underlying \( \mathbb{Z}_2 \) symmetry of the Hamiltonian (2.0.1) implies that such solutions must come in pairs related by \( G_{12}(\tau) \rightarrow -G_{12}(\tau) \) (in our numerical work we will typically exhibit only one of these two solutions). They correspond to working around the two inequivalent vacua, which we will call \( |0_+\rangle \) and \( |0_-\rangle \). They are distinguished by the sign of the expectation value of operator \( Q = i\chi^+_1\chi^+_2 \):

\[ \langle 0_+|Q|0_+\rangle = A , \quad \langle 0_-|Q|0_-\rangle = -A , \quad \langle 0_-|Q|0_+\rangle = 0 . \]  
(2.2.16)

\(^1\)These equations are also valid in the two-flavor tensor model (2.0.2), where \( G_{ab}(\tau) = \frac{1}{N^2}(\tau\psi^a_b(\tau)\psi^b_b(0)) \).
The unbroken symmetry $R$ in (2.2.9) implies

$$G_{12}(-\tau) = -G_{21}(\tau) = G_{12}(\tau), \quad G_{22}(\tau) = G_{11}(\tau), \quad (2.2.17)$$

and similarly for $\Sigma_{ab}$. Using these constraints, we obtain for the effective action

$$-\frac{\beta S_{\text{eff}}}{N_{\text{SYK}}} = \log \text{Pf}(\delta_{ab}\partial_\tau - \Sigma_{ab}) - \beta \int_0^\beta d\tau \left( \Sigma_{11}G_{11} + \Sigma_{12}G_{12} \right. \\
- \frac{J^2}{4} \left( (1 + 3\alpha^2)(G_{11}^4 + G_{12}^4) + 12\alpha(1 - \alpha)G_{11}^2G_{12}^2 \right). \quad (2.2.18)$$

The Schwinger Dyson equations become

$$\partial_\tau G_{11}(\tau) - \int d\tau' (\Sigma_{11}(\tau - \tau')G_{11}(\tau') - \Sigma_{12}(\tau - \tau')G_{12}(\tau')) = \delta(\tau),$$
$$\partial_\tau G_{12}(\tau) - \int d\tau' (\Sigma_{11}(\tau - \tau')G_{12}(\tau') + \Sigma_{12}(\tau - \tau')G_{11}(\tau')) = 0, \quad (2.2.19)$$

and

$$J^{-2}\Sigma_{11}(\tau) = (1 + 3\alpha^2)G_{11}^3(\tau) + 6\alpha(1 - \alpha)G_{11}(\tau)G_{12}^2(\tau),$$
$$J^{-2}\Sigma_{12}(\tau) = (1 + 3\alpha^2)G_{12}^3(\tau) + 6\alpha(1 - \alpha)G_{11}^2(\tau)G_{12}(\tau). \quad (2.2.20)$$

(2.2.19) may be written in momentum space as

$$G_{11}(\omega_n) = \frac{-i\omega_n - \Sigma_{11}(\omega_n)}{(-i\omega_n - \Sigma_{11})^2 + \Sigma_{12}^2}, \quad G_{12}(\omega_n) = \frac{\Sigma_{12}(\omega_n)}{(-i\omega_n - \Sigma_{11})^2 + \Sigma_{12}^2}. \quad (2.2.21)$$

These equations, together with (2.2.20), can be solved numerically using the method of weighted iterations used in [14].m To trigger the spontaneous symmetry breaking, we start our iteration process with a tiny non-zero $G_{12}(\tau)$ which is purely imaginary. If we are in the unbroken phase, after the iterations $G_{12}$ becomes zero; whereas if we are in the broken phase

)mIn this case we find it more convenient to use a slow decay rate on the weight $x$. 42
we find a non-zero purely imaginary solution for $G_{12}$.

Figure 2.9: Numerical solutions for $\alpha = -1, -0.5, -0.2$ and various values of $\beta J$.

The plots of $G_{11}$ and $iG_{12}$ for different values of $\alpha$ and $\beta J$ are shown in fig. 2.9. For each value of $\alpha$ between $-1$ and $0$ there are two phases. In the low temperature phase (large $\beta J$), there are three distinct solutions: two solutions with non-vanishing $iG_{12}$ related by

Figure 2.10: The expectation value of $Q$, i.e. $|G_{12}(0)|$, as a function of $\beta J$ for $\alpha = -0.5$. The region near $(\beta J)_{\text{crit}}$ is shown.

The plots of $G_{11}$ and $iG_{12}$ for different values of $\alpha$ and $\beta J$ are shown in fig. 2.9. For each value of $\alpha$ between $-1$ and $0$ there are two phases. In the low temperature phase (large $\beta J$), there are three distinct solutions: two solutions with non-vanishing $iG_{12}$ related by
$G_{12}(\tau) \to -G_{12}(\tau)$ and the one where $G_{12}(\tau) = 0$. The solutions with non-vanishing $iG_{12}$ are the ones with the lower free energy. As $\beta J$ decreases, $|G_{12}(\tau)|$ decreases everywhere for the non-trivial solution (see figure 2.9, 2.10), and at the critical value becomes exactly zero. For $\beta J < (\beta J)_{\text{crit}}$ the only possible solution is $G_{12}(\tau) = 0$. Thus, the $\mathbb{Z}_2$ symmetry is restored, and this is a second-order phase transition. The plot of $(\beta J)_{\text{crit}}$ vs. $\alpha$ is shown in figure 2.10; it blows up as $\alpha$ approaches zero from below.\(^n\)

Using the solutions of the Schwinger-Dyson equations we can numerically compute the large $N$ free energy

$$-\frac{\beta F}{N_{\text{SYK}}} = \log 2 + \frac{1}{2} \sum_{n=-\infty}^{+\infty} \log \left( \left( 1 + \frac{\Sigma_{11}(\omega_n)}{i\omega_n} \right)^2 - \frac{\Sigma_{12}^2(\omega_n)}{\omega_n^2} \right) + \frac{3}{4} \sum_{n=-\infty}^{+\infty} \left( \Sigma_{11}(\omega_n)G_{11}(\omega_n) - \Sigma_{12}(\omega_n)G_{12}(\omega_n) \right), \quad (2.2.22)$$

where the sum $\sum_n \log(-i\omega_n)$ is replaced by $\log(2)$. The energy can be computed with the formula

$$\frac{E}{N_{\text{SYK}}} = \frac{1}{2\beta} \sum_{n=-\infty}^{+\infty} \left( \Sigma_{11}(\omega_n)G_{11}(\omega_n) - \Sigma_{12}(\omega_n)G_{12}(\omega_n) \right) \quad (2.2.23)$$

and at low temperatures it should converge to the energy of the ground state $E_0$ divided by $N_{\text{SYK}}$.

Now one can compare the free energy in the symmetry broken phase, $F_{G_{12} \neq 0}$ with that of the symmetry unbroken phase, $F_{G_{12} = 0}$. In particular, the free energy of the latter is simply twice that of a single SYK with a rescaling $J \to \sqrt{1 + 3\alpha^2} J$. As a check, in such cases one can consider a large $q$ expansion [14, 72],

$$\beta F_{G_{12} = 0}^q = -\log 2 - \frac{1}{q^2} 2\pi \nu \left( \tan \frac{\pi \nu}{2} - \frac{\pi \nu}{4} \right) - \frac{1}{q^3} 2\pi \nu \left( \pi \nu - 2 \tan \pi \nu \left( 1 - \frac{\pi^2 \nu^2}{12} \right) \right) + \ldots, \quad (2.2.24)$$

\(^n\)We note that this function does not have a vanishing derivative at the self-dual value of $\alpha = -1$. Had we plotted the critical value of $\beta \tilde{J} = \beta J \sqrt{\lvert \alpha \rvert}$, this derivative would vanish but the plot would not be monotonic.
where \( \beta J \sqrt{(1 + 3\alpha^2)^2} \approx q = \frac{\pi \mu}{\cos \frac{\pi}{2}} \). The free energy of the symmetry unbroken phase \( F_{G_{12}=0} \) is seen to agree well numerically with \( F_{G_{12}=0}^{\text{true}} \).

In figure 2.12 we plot for \( \alpha = -1 \) the free energy of the symmetry broken phase (2.2.22) as a function of \( \beta J \) and compare it with that of the unbroken phase, obtained by setting \( G_{12} = 0 \) in the SD equations (2.2.19) and (2.2.20). We also show the entropy as a function of \( \beta J \). The plot shows a clear second order phase transition at \( (\beta J)_{\text{crit}} \approx 2.87 \), and the derivative of the entropy is discontinuous. We will systematically study the critical exponents in future work.

We notice that at sufficiently large \( \beta J \), there is a range of \( \tau \) where both \( iG_{12}(\tau) \) and
Figure 2.13: Large $N$ free energies at fixed $\beta$ and $J$. We take $\beta = 5$, $J = 1$, and decrease $\alpha$. We observe also a second order phase transition.

$G_{11}(\tau)$ decay exponentially and share the same decay rate. To explain this fact, let us study the $T = 0$ case and insert the complete set of states

$$G_{11}(\tau) = \langle 0_+| e^{-H\tau} \chi_1^1(0)e^{H\tau}|n\rangle \langle n| \chi_1^1(0)|1_+\rangle .$$

\hspace{1cm} (2.2.25)

For large $\tau$ the sum is dominated by the lowest excited state, and we find

$$G_{11}(\tau) \rightarrow e^{-(E_1-E_0)\tau} \langle 0_+| \chi_1^1(0)|1\rangle \langle 1| \chi_1^1(0)|0_+\rangle .$$

\hspace{1cm} (2.2.26)

Similarly, we find that the large $\tau$ behavior of $G_{12}$ is

$$G_{12}(\tau) \rightarrow e^{-(E_1-E_0)\tau} \langle 0_+| \chi_2^1(0)|1\rangle \langle 1| \chi_2^1(0)|0_+\rangle .$$

\hspace{1cm} (2.2.27)

Thus the universal decay rate among correlators signifies a mass gap in the spectrum.

In the work of Maldacena and Qi [60] the functions $G_{11}$ and $G_{12}$ were also found to be exponentially decreasing for sufficiently large $\beta J$. In fig. 2.14 we exhibit superimposed plots of the low temperature solutions to our system of equations and those from [60], with parameters chosen so that the solutions are close to one another for most of the range. We observe a difference in the behavior of $iG_{12}(\tau)$ and $iG_{LR}(\tau)$ at small $\tau$: in our case the function is smooth with a vanishing derivative at $\tau = 0$, while in [60] its derivative is
discontinuous at $\tau = 0$; this is due to the fact that their Hamiltonian includes a quadratic term.

\[
\beta J = 100, \quad \beta \mu = 12, \quad \alpha = -0.15
\]

Figure 2.14: Plot of solutions $G_{LL}$ and $iG_{LR}$ for the model in [60] superimposed with $G_{11}$ and $iG_{12}$. Parameters chosen so that the solutions are close for most of the range of $\theta$.

Figure 2.15: The expectation value of $Q$, i.e. $|G_{12}(0)|$, as a function of $\alpha$ for $\beta J = 5000$.

We can also study what happens at low temperatures (large $\beta J$) as a function of $\alpha$. In figure 2.15 we plot $iG_{12}(0)$, which is the expectation value of the order parameter $Q$ for a large $\beta J$. This quantity becomes small as $\alpha$ is increased towards zero. In figure 2.16 we plot the large $N_{\text{SYK}}$ limit of the energy gap $E_{\text{gap}}$ divided by $J N_{\text{SYK}}$, calculated from the exponential decay of the Green functions. We also plot the ground state energy $E_0$ divided by $J N_{\text{SYK}}$ calculated using (2.2.23). In the next section these results will be compared with exact diagonalizations for finite $N_{\text{SYK}}$. 

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Figure 2.16: Right: the value of $E_0/(JN_{\text{SYK}})$ as a function of $\alpha$. Both graphs are approximately linear in $\alpha$ for $\alpha$ not too small. Left: the large $N_{\text{SYK}}$ energy gap in the spectrum, computed from the exponential decay of the Green functions.

2.2.3 Exact diagonalization for finite $N_{\text{SYK}}$

In this section we present numerical results for the spectra of two coupled SYK models with Hamiltonian (2.0.1). We first check that the results from exact diagonalizations agree well with expectations: the spectrum for $\alpha = 0$ and $N_{\text{SYK}} = 30$, and the ground state energy of $\alpha = -1$ for various $N_{\text{SYK}}$ concur well with analytical arguments, and with the results from 2.2.2. Then we present our results on the energy gap and broken symmetry. The biggest number we are able to access via exact diagonalization of the coupled SYK models is $N_{\text{SYK}} = 16$. In this case the discrete symmetry (2.3.5) is not anomalous, and the ground state is non-degenerate. However, for $-1 \leq \alpha < 0$ we observe a nearby excited state followed by a gap. We will interpret this as indication of spontaneous symmetry breaking. We will also present spectra for $N_{\text{SYK}} = 15$, where the discrete symmetry (2.3.5) is anomalous, so that the states are doubly degenerate. There is again a gap in the spectrum present for $-1 \leq \alpha < 0$. Furthermore, we present numerical results on the VEV of operator $i\chi_1^i \chi_2^j$ for $N_{\text{SYK}} = 14$, which demonstrates that it is non-vanishing for $-1 \leq \alpha < 0$.

First, let us consider $\alpha = 0$, where we find the spectrum of two SYK model with the same random couplings. The density of states for this model is simply given by the convolution
Figure 2.17: Left: The energy spectrum for $\alpha = 0$, i.e. for two decoupled SYK models, for a single sampling of $N_{\text{SYK}} = 30$. Right: The same spectrum magnified near the edge.

of that of the single SYK model:

$$\rho_{\text{double}}(E) = \int de \rho(e) \rho(E - e)$$ \hspace{1cm} (2.2.28)

This in particular helps us determine the behavior of $\rho_{\text{double}}(E)$ near the ground state. Shifting the energy so that the ground state is at zero, we know that $\rho(E) \rightarrow A\sqrt{E}$ for small $E$. Therefore, for small $E$

$$\rho_{\text{double}}(E) \rightarrow A^2 \int_0^E de \sqrt{e(E - e)} = \frac{\pi A^2 E^2}{8}. \hspace{1cm} (2.2.29)$$

The numerical density of states, shown in figure 2.17 for $N_{\text{SYK}} = 30$, is in good agreement with the $E^2$ dependence near the ground state.

Let us proceed to the spectra for non-vanishing values of $\alpha$. In figs. 2.18, 2.20 we plot the spectra of energy divided by $J$ for $\alpha = -1, -1/2, 1/3$ and different values of $N_{\text{SYK}}$. These energy distributions have interesting and unusual shapes. For the special values $\alpha = -1$ and $1/3$ we observe large numbers of states with $E = 0$; this creates the zero-energy peaks seen in the graphs. For $\alpha = -1$ and odd $N_{\text{SYK}}$ we find that the $E = 0$ peak is separated by gaps.

\footnote{We thank D. Stanford for a useful discussion about this.}
from the remaining states, but for even $N_{\text{SYK}}$ it is not.

Figure 2.18: The spectrum for a single realization with $N_{\text{SYK}} = 15, 16$ and $\alpha = -1, -0.5$. For $\alpha = -1$, the spectrum exhibits a gap near $E = 0$ when $N_{\text{SYK}}$ is odd and a large number of states with $E = 0$. 
For $N_{\text{SYK}} = 15$, due to the anomaly in particle-hole symmetry, there are two degenerate ground states, see fig. 2.18. In fact, each energy level is doubly degenerate. Nevertheless, for $-1 \leq \alpha < 0$ we observe a gap between the lowest energy level and the next one, as expected. On the other hand, for $N_{\text{SYK}} = 16$ there is no exact degeneracy of the ground state, but the first gap is very small, indicating a tendency towards spontaneous symmetry breaking at large $N_{\text{SYK}}$. We show the $N_{\text{SYK}} = 16$ spectra for $\alpha = -1$ and $\alpha = -0.5$ in fig. 2.18. In both cases, for a typical sampling of the coupling constants $J_{ijkl}$ we observe two closely spaced states followed by a visible gap. For large $N_{\text{SYK}}$ the energy gap between the two lowest states is expected to decrease exponentially:

$$- \log \frac{E_1 - E_0}{J} \sim N_{\text{SYK}}.$$  \hspace{1cm} (2.2.30)

We check this in figure 2.19 at $\alpha = -1$.

Figure 2.19: The size of $\log \frac{E_1 - E_0}{J} = \log \Delta E$ at $N_{\text{SYK}} = 8, 12, 16, 20, 24$, at $\alpha = -1$(The number of samples are: 33886, 20326, 4080, 860, 30.). The error bars are produced with a 95% confidence interval. The dashed line is a linear fit. To make the numerical computation feasible, we used the $Z_4$ symmetry to reduce the size of the Hilbert space. The $Z_4$ charge can be conveniently represented by the $U(1)$ charge mod 4. For $N_{\text{SYK}} = 0 \mod 8$, the two closely degenerate states are both in the neutral sector. For $N_{\text{SYK}} = 4 \mod 8$, they are in the 2 mod 4 sector.

For $\alpha \geq 0$ the low-lying spectrum is different – we observe many closely spaced low-lying states without large gaps, similarly to the standard SYK spectrum.

If we instead adopt the Maldacena-Qi Hamiltonian with a quadratic coupling which breaks the time-reversal symmetry explicitly, there is no such double degeneracy.
Figure 2.20: The spectrum for a single realization for $N_{\text{SYK}} = 16$ and $\alpha = 1/3$.

In fig. 2.21 we plot the ground state energy for $\alpha = -1$ and various values of $N_{\text{SYK}}$. The plot is approximately linear, and the fit gives

$$E_{0}^{\alpha=-1} = -0.284N_{\text{SYK}} + 0.379.$$ \hspace{1cm} (2.2.31)

The limiting value $E_{0}^{\alpha=-1}/N_{\text{SYK}} = -0.284$ is in good agreement with the result found from Schwinger-Dyson equations; see fig. 2.16.

Figure 2.21: Left: The ground state energy for $\alpha = -1$ and $N_{\text{SYK}} = 10, 11, \ldots, 16$ (The number of samples are: 250000, 120000, 50000, 5000, 5000, 2000, 500). The linear fit is shown by dashed line. Right: The energy gap between second and third states as a function of $\alpha$ for a single realization of random couplings at $N_{\text{SYK}} = 16$.

In figure 2.21 we exhibit the energy gap between second and third states as a function of $\alpha$. As $\alpha$ is increased from $-1$ to $0$, the gap decreases as expected.

Exact diagonalizations also provide support for the statement that the fermion number $Q$ acquires a vacuum expectation value for $-1 \leq \alpha < 0$. For $N_{\text{SYK}}$ not divisible by 4, there are
two ground states $|0_\pm\rangle$ which map into each other under the symmetry operator $P$. This can be viewed as anomalous breaking of the time-reversal $\mathbb{Z}_2$ symmetry (2.2.10) which occurs for a finite number of degrees of freedom [54–56,58,59]. In figure 2.22 the vacuum expectation value as a function of $\alpha$ is plotted for $N_{\text{SYK}} = 14$. This is the finite $N_{\text{SYK}}$ analogue of fig. 2.15, where the large $N_{\text{SYK}}$ limit of the condensate is plotted. We also note the qualitative similarity of the plot 2.22 and that of the imaginary part of the scaling dimension of $Q$ in fig. 2.8.

![Figure 2.22](image)

**Figure 2.22:** The expectation value $\langle 0_+ | Q | 0_+ \rangle$ as a function of $\alpha$ for a single realization of random couplings at $N_{\text{SYK}} = 14$.

### 2.3 Appendix: More on the discrete symmetries

The model (2.0.1) has the anti-unitary particle-hole $\mathbb{Z}_2$ symmetry generated by (2.2.10). The operator $K$ is defined to take $z \rightarrow \bar{z}$, $z \in \mathbb{C}$ but acts as the identity on $\psi$ or $\bar{\psi}$. It may be identified as a kind of time-reversal generator which satisfies $K^2 = 1$ [54–56]. It acts by

\[
KiK = -i, \quad K\chi^i_1K = \chi^i_1, \quad K\chi^i_2K = -\chi^i_2, \quad (2.3.1)
\]

and therefore, satisfies

\[
[K, H] = [K, Q] = 0. \quad (2.3.2)
\]
Note that although $K$ can be anomalous, $K$ is unbroken as it does not change the sign of $Q$. Another unbroken symmetry is the $\frac{\pi}{2}$ rotation between $\chi^i_1$ and $\chi^i_2$.

$$R = (-1)^{N_{SYK}/4}2^{-N_{SYK}^2} \prod_i (1 - 2\chi^i_1\chi^i_2).$$  \hspace{1cm} (2.3.3)

It satisfies

$$RR^\dagger = 1, \quad R\chi^i_1 R^\dagger = \chi^i_2, \quad R\chi^i_2 R^\dagger = -\chi^i_1, \quad R^4 = 1.$$  \hspace{1cm} (2.3.4)

Note $R^2 = (-1)^F$. There are also various reflection $\mathbb{Z}_2$ symmetries that are spontaneously broken by the VEV of $Q$. In particular, we have the reflection symmetry:

$$P = \begin{cases} 
(-1)^{N_{SYK}(N_{SYK}-1)/4}2^{N_{SYK}/2} \prod_{i=1}^{N_{SYK}} \chi^i_1 & \text{if } N_{SYK} = 2k, k \in \mathbb{Z} \\
(-1)^{N_{SYK}(N_{SYK}-1)/4}2^{N_{SYK}/2} \prod_{i=1}^{N_{SYK}} \chi^i_2 & \text{if } N_{SYK} = 2k + 1, k \in \mathbb{Z},
\end{cases}$$  \hspace{1cm} (2.3.5)

such that

$$PP^\dagger = 1, \quad P\chi^i_1 P^\dagger = -\chi^i_1, \quad P\chi^i_2 P^\dagger = \chi^i_2, \quad P^2 = 1.$$  \hspace{1cm} (2.3.6)

In fact, $R$, $P$, and $K$ are enough to generate all discrete symmetries of the model (2.0.1). In particular, all the unitary discrete symmetries form $D_4$, the dihedral group of order 8. To see this, it’s enough to check that the group presentation: $R^4 = P^2 = (RP)^2 = 1$. The remaining reflections can be identified with $RP, R^2P$ and $R^3P$. For a given unitary symmetry we can compose it with $K$ to obtain an anti-unitary one.

In our case, when $N_{SYK} \to \infty$, although multiple $\mathbb{Z}_2$ symmetries are spontaneously broken, we only expect a two-fold ground state degeneracy. In fact, any two broken symmetries that can be related by an unbroken symmetry do not produce any extra ground state degeneracy. To see this, consider for example the reflection symmetry $RP$. Since $R$ is unbroken, we may assume $R\ket{0} = \ket{0}$ without losing of generality. Then $RP\ket{0} = RPR\ket{0} = P\ket{0}$.

At finite $N_{SYK}$, however, certain discrete symmetry can be anomalous and is responsible
for an exact two fold degeneracy for certain \( N_{\text{SYK}} \). For example, the particle-hole symmetry \( \mathcal{P} \sim K P \) acts on the fermions as

\[
\mathcal{P} \psi^j \mathcal{P} = \eta \bar{\psi}^j, \quad \mathcal{P} \bar{\psi}^j \mathcal{P} = \eta \psi^j, \quad \eta = (-1)^{(N_{\text{SYK}}+2)(N_{\text{SYK}}-1)/2}.
\]  

(2.3.7)

The fermion number operator (2.2.6) is odd under this symmetry:

\[
\mathcal{P} Q \mathcal{P} = -\mathcal{P}^2 Q .
\]  

(2.3.8)

When \( N_{\text{SYK}} \) is not divisible by 4, there are two degenerate ground states \(|0_\pm\rangle\), and the symmetry generator \( \mathcal{P} \) maps them into each other [54–59]:

\[
\mathcal{P} |0_+\rangle = (-1)^{N_{\text{SYK}}(N_{\text{SYK}}-1)/4} |0_-\rangle, \quad \mathcal{P} |0_-\rangle = (-1)^{N_{\text{SYK}}(N_{\text{SYK}}-1)/4} |0_+\rangle.
\]  

(2.3.9)

In this case we can say that the particle-hole symmetry is anomalous.
Chapter 3

The spontaneous breaking of $U(1)$ symmetry

In chapter 2 it is shown that when two tensor or SYK models are coupled by certain quartic interactions with a coefficient $\alpha$, a line of fixed points emerge when $\alpha$ is positive in its admissible range, while a gapped $Z_2$ symmetry breaking phase appears when $\alpha$ is negative.

In this chapter we make further progress in this direction by obtaining similar coupled models where a $U(1)$ symmetry is broken spontaneously in the large $N$ limit. Our starting point is the complex SYK model [23, 25, 39, 73] (see also the earlier work [74, 75]), which has a $U(1)$ global symmetry. When two such models are coupled together by a quartic interaction preserving the $U(1) \times U(1)$ symmetry,

$$H = \sum_{i,j,k,l=1}^{N} J_{ij,kl} \left( c_{1i}^{\dagger} c_{1j}^{\dagger} c_{1k} c_{1l} + c_{2i}^{\dagger} c_{2j}^{\dagger} c_{2k} c_{2l} + 8\alpha c_{1i}^{\dagger} c_{2j}^{\dagger} c_{2k} c_{1l} \right),$$

we find that it is possible to break one of the $U(1)$ symmetries spontaneously. The phase where the $U(1)$ symmetry is broken by a VEV of operator $c_{1i}^{\dagger} c_{2i}$ is found for $\alpha < 0$ and $\alpha > 1$. In contrast with the breaking of discrete symmetry in the coupled Majorana SYK model [1], there is no gap in the full large $N$ spectrum due to the Nambu-Goldstone phenomenon. It
manifests itself in splittings of order $1/N$ between the lowest states in different charge sectors. However, some specific charge sectors exhibit gaps of order 1 above the ground state.

We also exhibit a tensor counterpart of the coupled random model (3.0.1) which consists of two coupled complex tensor models. The basic such model with $SU(N)^2 \times O(N) \times U(1)$ symmetry was introduced in [15], and the two are coupled by an interaction which preserves the $SU(N)^2 \times O(N) \times U(1)^2$ symmetry.\footnote{The meaning of $N$ in the tensor models is different from that in the SYK models.}

At the special coupling $\alpha = 1/4$, the $U(1) \times U(1)$ symmetry is enhanced to $U(2) \sim U(1) \times SU(2)$, and the Hamiltonian (3.0.1) may be written compactly as

$$H_{U(2)} = \sum_{i,j,k,l=1}^{N} J_{ij,kl} c_{\sigma i}^\dagger c_{\sigma j}^\dagger c_{\sigma' k} c_{\sigma' l},$$

where there is a sum over $\sigma, \sigma' = 1, 2$. This is equal to the quartic term in the model of [76], which was argued to provide a description of quantum dots with irregular boundaries. In (3.0.2) the $U(1)$ is the usual charge symmetry, while the enhanced $SU(2)$ symmetry models the physical spin; we may think of $\sigma$ as labeling the two spin states, up and down.

We note that some results on spontaneous $U(1)$ symmetry breaking in models with random couplings have already appeared in the literature [77–84]. For example, toy models of superconductivity introduced in [78,79,84] include random Yukawa interactions of fermion-phonon type.

Other recently introduced models [77,80,81] include random quartic couplings, as well as the non-random double-trace operator $OO^\dagger$, where $O$ is a "Cooper pair operator" $O \sim c_{i\uparrow} c_{i\downarrow}$. The models we study in this chapter are somewhat different, and they appear to be the first examples of manifestly melonic theories where the spontaneous breaking of $U(1)$ symmetry can be established through analysis of the exact large $N$ Dyson-Schwinger equations.

The structure of the chapter is as follows. In section 3.1 we introduce some melonic models with $U(1) \times U(1)$ symmetry. They include a pair of coupled complex SYK models...
with Hamiltonian (3.0.1), as well as the tensor counterpart of this model with Hamiltonian (3.1.11). In section 3.2 we discuss the symmetric saddle point of the large $N$ effective action, as well as fluctuations around it. There is a range $0 \leq \alpha \leq 1$ where the symmetric saddle point is stable, while outside this fixed line a fermion bilinear operator, $c_{1i}^{\dagger}c_{2i}$, acquires a complex scaling dimension. In section 4.1.19 we find a more general solution of the Dyson-Schwinger equations, which contains the off-diagonal Green’s function $G_{12}$. It is stable outside the fixed line and indicates that the operator $c_{1i}^{\dagger}c_{2i}$ acquires an expectation value. This phase of the theory is characterized by the exponential fall-off of Green’s functions at low temperatures. In section 3.4 we discuss the low-energy effective action in this phase and calculate the compressibility for the broken $U(1)$ degree of freedom. In section 3.5 we support some of these results by Exact Diagonalizations at accessible values of $N$. Extrapolating the ground state energies and compressibilities to large $N$, we obtain good agreement with some of the results obtained using the DS equations. In section 3.6 we present results for compressibilities at the special value $\alpha = 1/4$ where the model has $U(2)$ symmetry. Some additional details can be found in the Appendices.

3.1 Melonic models with $U(1) \times U(1)$ symmetry

In this section we introduce some melonic models with quartic Hamiltonians, which possess $U(1) \times U(1)$ symmetry. The first model with Hamiltonian (3.0.1) consists of two copies of complex SYK model with a marginal $U(1) \times U(1)$ preserving interaction containing a dimensionless coupling, $\alpha$. We also formulate its tensor counterpart which has $SU(N)^2 \times O(N) \times U(1)^2$ symmetry; it has the same Dyson-Schwinger equations as the random model. 

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b The new paper [85] by S. Sahoo et al. where the model (3.0.1) is also studied, with results similar to some of ours.
3.1.1 Two coupled complex SYK models

Consider two sets of \( N \) complex fermions, \( c_{\sigma i} \), where \( \sigma = 1, 2 \) and \( i = 1, \ldots, N \):

\[
\{ c_{\sigma i}^\dagger, c_{\sigma' j} \} = \delta_{\sigma \sigma'} \delta_{ij} . \tag{3.1.1}
\]

The Hamiltonian coupling them is (3.0.1), where \( J_{ij,kl} \) is the random Gaussian complex tensor with zero mean \( \overline{J_{ij,kl}} = 0 \); it satisfies \( J_{ij,kl} = J^*_{kl,ij} \) in order for the Hamiltonian to be Hermitian. We also assume anti-symmetry in the first and second pairs of indices: \( J_{ij,kl} = -J_{ji,kl} = -J_{ij,lk} \). The variance is \( |J_{ij,kl}|^2 = J^2/(2N)^3 \).

So far the definition of the random tensor \( J_{ij,kl} \) is incomplete. In fact, there is some freedom in its definition [86] even for the single complex SYK model. In this chapter we will not use this freedom and will adopt the following minimal approach. We decompose \( J_{ij,kl} \) as

\[
J_{ij,kl} = \frac{1}{4} (T_{ij,kl} + T^*_{kl,ij}) ,
\]

where \( T_{ij,kl} \) is antisymmetric in the first and second pairs of indices and has no other symmetries. We then treat \( T_{ij,kl} \) as \( N^2(N-1)^2/4 \) independent complex Gaussian random variables, so \( \overline{T_{ij,kl}} = 0 \) and \( |T_{ij,kl}|^2 = J^2/N^3 \).

The Hamiltonian (3.0.1) has two \( U(1) \) symmetries,

\[
U(1)_+ : \quad c_1 \rightarrow e^{i\phi} c_1 , \quad c_2 \rightarrow e^{i\phi} c_2 ;
\]

\[
U(1)_- : \quad c_1 \rightarrow e^{-i\phi} c_1 , \quad c_2 \rightarrow e^{-i\phi} c_2 . \tag{3.1.2}
\]

The corresponding conserved charges are

\[
Q_\pm = Q_1 \pm Q_2 = \frac{1}{2} \sum_{i=1}^{N} \left( \left[ c_{1i}^\dagger, c_{1i} \right] \pm \left[ c_{2i}^\dagger, c_{2i} \right] \right) . \tag{3.1.3}
\]

Both \( Q_+ \) and \( Q_- \) take integer values ranging from \(-N\) to \( N \), with the constraint that \( Q_+ + Q_- \) is even for \( N \) even and odd for \( N \) odd.\(^{\circ}\)

\(^{\circ}\)We may consider a variant of the model where the \( U(1)_+ \) symmetry is gauged; in this case we have to restrict the Hilbert space to the sector with \( Q_+ = 0 \).
The Hamiltonian also has the $Z_4$ symmetry

\[ c_{1i} \rightarrow c_{2i}, \quad c_{2i} \rightarrow -c_{1i}, \quad (3.1.4) \]

which is analogous to the $Z_4$ symmetry which played an important role in [1]. Another important symmetry is the particle-hole symmetry

\[ c_{1i} \leftrightarrow c_{1i}^\dagger, \quad c_{2i} \leftrightarrow c_{2i}^\dagger, \quad J_{ij,kl} \rightarrow J_{ij,kl}^*, \quad (3.1.5) \]

In order to make the Hamiltonian invariant under this symmetry for general $\alpha$, we have to add to it certain quadratic and c-number terms which are exhibited in (3.7.4).\(^d\)

Note that, since the random coupling $J_{ij,kl}$ is complex, the $U(1)_+$ and $U(1)_-$ are on a different footing: the charge conjugation acting on the second flavor $c_{2i}$,

\[ C_2^i c_{2i} C_2 = c_{2i}^\dagger \quad (3.1.6) \]

is not a symmetry of the Hamiltonian (3.0.1). The $U(1)_+$ is the overall charge symmetry, while the “axial” symmetry $U(1)_-$ may be thought of as a spatial rotation around the third axis. We will show that, for $\alpha < 0$, the $U(1)_-$ may be broken spontaneously in the large $N$ limit, but the charge symmetry $U(1)_+$ remains unbroken. Holographically, the $U(1)_-$ has a simple physical meaning: a holographic state charged under $U(1)_-$ corresponds to bulk solutions with an electric field turned on.

Using the standard procedure for integrating over disorder and introducing bilocal fields,

\[ G_{\sigma\sigma'}(\tau_1, \tau_2) = \frac{1}{N} \langle T c_{\sigma i}(\tau_1) c_{\sigma i}^\dagger(\tau_2) \rangle, \quad (3.1.7) \]

\(^d\)Note that, together with the symmetry which exchanges $c_1^i$ and $c_2^i$, the unitary discrete symmetries of (3.0.1) are $D_4 \times Z_2$. 
and \( \Sigma_{\sigma\sigma'}(\tau_1, \tau_2) \), we write down the effective action

\[
I = - \log \det(\delta'(\tau_1)\delta_{\sigma\sigma'} - \Sigma_{\sigma\sigma'}(\tau_1, \tau_2)) - \int d\tau_1 d\tau_2 \Sigma_{\sigma\sigma'}(\tau_1, \tau_2) G_{\sigma'\sigma}(\tau_2, \tau_1) - \frac{J^2}{4} \int d\tau_1 d\tau_2 V(G_{\sigma\sigma'}) ,
\]

\[
V(G_{\sigma\sigma'}) = G_{11}^2(\tau_1, \tau_2)G_{11}^2(\tau_2, \tau_1) + G_{22}^2(\tau_1, \tau_2)G_{22}^2(\tau_2, \tau_1) + 2G_{12}^2(\tau_1, \tau_2)G_{21}^2(\tau_2, \tau_1) + 16\alpha(G_{11}(\tau_1, \tau_2)G_{11}(\tau_2, \tau_1) + G_{22}(\tau_1, \tau_2)G_{22}(\tau_2, \tau_1))G_{12}(\tau_1, \tau_2)G_{21}(\tau_2, \tau_1) + 16\alpha^2(G_{11}(\tau_1, \tau_2)G_{22}(\tau_2, \tau_1) + G_{12}(\tau_1, \tau_2)G_{21}(\tau_1, \tau_2))(G_{11}(\tau_2, \tau_1)G_{22}(\tau_2, \tau_1) + G_{12}(\tau_2, \tau_1)G_{21}(\tau_2, \tau_1)) .
\]

(3.1.8)

For \( \alpha = 1/4 \) this can be nicely written as

\[
V(G_{\sigma\sigma'}) = \frac{1}{2} \text{Tr}(G(\tau_1, \tau_2)G(\tau_2, \tau_1))^2 + \frac{1}{2} \text{Tr}(G(\tau_1, \tau_2)G(\tau_2, \tau_1)G(\tau_1, \tau_2)G(\tau_2, \tau_1)) ,
\]

(3.1.9)

where \( G(\tau_1, \tau_2) \) is a \( 2 \times 2 \) matrix with elements \( G_{\sigma\sigma'}(\tau_1, \tau_2) \). So we can clearly see that \( V(G_{\sigma\sigma'}) \) is invariant under the global \( \text{SU}(2) \) transformations \( G(\tau_1, \tau_2) \rightarrow U^\dagger G(\tau_1, \tau_2)U \).

### 3.1.2 Tensor counterpart of the random model

Let us recall that the tensor counterpart of the standard complex SYK model [25, 39] is given by the tensor model with Hamiltonian [12, 15]

\[
h = g\bar{\psi}^{a_1b_1c_1} \psi^{a_2b_2c_2} \bar{\psi}^{a_1b_1c_2} \psi^{a_2b_2c_1} .
\]

(3.1.10)

The tensor indices range from 1 to \( N \), so that the model contains \( N^3 \) fermions, and the dimension of its Hilbert space is \( 2^{N^3} \). The model has \( \text{SU}(N)^2 \times \text{O}(N) \times \text{U}(1) \) symmetry: the \( \text{O}(N) \) symmetry acts on the second index of the tensor, while the two \( \text{SU}(N) \) symmetries act on the first and third indices, respectively. Exchanging the two \( \text{SU}(N) \) groups changes \( h \rightarrow -h \).

Now we need to similarly determine the tensor counterpart of two coupled cSYK models (3.0.1). As we show in Appendix 3.9, the same Dyson-Schwinger equations as for this random
model follow from the coupled tensor model with \( SU(N)^2 \times O(N) \times U(1)^2 \) symmetry, which has the Hamiltonian

\[
H_{\text{tensor}} = \frac{g}{2} \left( \bar{\psi}_1^{a_1 b_1 c_1} \psi_1^{a_2 b_2 c_2} + \bar{\psi}_2^{a_1 b_1 c_1} \psi_2^{a_2 b_2 c_2} - 2 \bar{\psi}_1^{a_1 b_1 c_1} \psi_2^{a_2 b_2 c_2} \right) .
\]

\[
+ 4\alpha \left( \bar{\psi}_1^{a_1 b_1 c_1} \psi_2^{a_2 b_2 c_2} - \bar{\psi}_2^{a_1 b_1 c_1} \psi_1^{a_2 b_2 c_2} \right) .
\]

(3.1.11)

Under interchange of the two \( SU(N) \) groups the Hamiltonian changes sign, and we have chosen the coupling term multiplied by \( \alpha \) to preserve this discrete symmetry. The \( U(1) \times U(1) \) symmetry acts analogously to that in the random model,

\[
U(1)_+ : \quad \psi_1^{abc} \rightarrow e^{i\phi} \psi_1^{abc}, \quad \psi_2^{abc} \rightarrow e^{i\phi} \psi_2^{abc};
\]

\[
U(1)_- : \quad \psi_1^{abc} \rightarrow e^{-i\phi} \psi_1^{abc}, \quad \psi_2^{abc} \rightarrow e^{-i\phi} \psi_2^{abc}.
\]

(3.1.12)

The Hamiltonian is also symmetric under the \( \pi/2 \) rotation \( \psi_1^{abc} \rightarrow \psi_2^{abc}, \quad \psi_2^{abc} \rightarrow -\psi_1^{abc} \).

In the tensor model (3.1.11) we may gauge the non-abelian symmetry \( SU(N)^2 \times O(N) \), restricting the states and operators to the sector invariant under this symmetry. Furthermore, as in the random counterpart (3.0.1), it is possible to gauge the \( U(1)_+ \) symmetry.

For \( \alpha = 1/4 \), the symmetry is enhanced to \( SU(N)^2 \times O(N) \times U(2) \), and the Hamiltonian may be written as

\[
H_{\text{tensor}} = \frac{g}{4} \left( \bar{\psi}_1^{a_1 b_1 c_1} \psi_1^{a_2 b_2 c_2} + \bar{\psi}_2^{a_1 b_1 c_1} \psi_2^{a_2 b_2 c_2} - 2 \bar{\psi}_1^{a_1 b_1 c_1} \psi_2^{a_2 b_2 c_2} \right) .
\]

(3.1.13)

For \( \alpha = 0 \), the Hamiltonian (3.1.11) becomes a sum of two Hamiltonians (3.1.10). In the tensor model (3.1.11) the gauged \( SU(N) \) symmetries forbid correlators of the form \( \langle \psi_1^{abc}(t) \psi_2^{abc}(0) \rangle \), and the corresponding operators \( \psi_1^{abc} \partial_t^m \psi_1^{abc} \) are not allowed (in the random model, these operators do not receive ladder corrections). The symmetries do allow correlators of the form \( \bar{\psi}_1^{abc} \partial_t^m \psi_1^{abc} \), and their large \( N \) scaling dimensions are non-trivial. We will determine their values as functions of \( \alpha \) in the next section, and show that one of them
is complex for $\alpha < 0$ and $\alpha > 1$.

### 3.2 Scaling Dimensions of Fermion Bilinears

First let us study the large $N$ saddle point where

$$G_{12} = G_{21} = 0 \quad \Sigma_{12} = \Sigma_{21} = 0, \quad (3.2.1)$$

so that the $U(1)_+ \times U(1)_-$ symmetry is preserved. Next it is reasonable to assume that $G_{11}(\tau_1, \tau_2) = G_{22}(\tau_1, \tau_2) = G(\tau_{12})$, where $G(\tau)$ is the particle-hole symmetric Green’s function, so $G(-\tau) = -G(\tau)$. And we obtain

$$\partial_\tau G(\tau) - \int d\tau' \Sigma(\tau - \tau') G(\tau') = \delta(\tau),$$

$$\Sigma(\tau) = J^2(1 + 8\alpha^2) G^3(\tau), \quad (3.2.2)$$

which is the standard SYK Dyson-Schwinger (DS) equations with $J' = J\sqrt{1 + 8\alpha^2}$. We will see that this diagonal saddle point describes the theory in the range $0 \leq \alpha \leq 1$, where various large $N$ quantities are related to those in the $\alpha = 0$ theory by the rescaling of $J$. For example, the ground state energy is

$$E_0(\alpha) = 2E_0^{cSYK} \sqrt{1 + 8\alpha^2} \approx -0.1624NJ\sqrt{1 + 8\alpha^2}. \quad (3.2.3)$$

Now we consider the bilinear spectrum at the nearly conformal saddle 3.2.2. They can be obtained by considering the melonic Bethe-Salpeter equations for the three point functions. Due to $U(1)_+ \times U(1)_-$ symmetry, we can separate the computations in terms of

$$v_{\sigma\sigma'}^{0,2(\sigma-\sigma')}(\tau, 0, \infty) = \langle c_{\sigma 1}(\tau) c_{\sigma' 1}(0) \mathcal{O}_h^{0,2(\sigma-\sigma')}(\infty) \rangle, \quad (3.2.4)$$
between elementary fermions $c_{\sigma^i}^\dagger, c_{\sigma^i}$ and a primary operator $O^{0,2(\sigma-\sigma')}_h$ with dimension $h$ and $U(1)_+ \times U(1)_-$ charge $(0, 2(\sigma - \sigma'))$. Note that operators with non-zero $U(1)_+$ charge do not receive ladder correction in the large $N$ limit due to $J_{ij,kl}$ being complex.

It is convenient to write down the explicit forms of the primary operators $\{O^{0,0}_m, O^{0,0}_m, O^{0,2}_m, O^{0,-2}_m\}$, where

$$O^{0,0}_m = c_{1i}^\dagger \partial^m c_{1i} \pm c_{2i}^\dagger \partial^m c_{2i}, \quad O^{0,2}_m = c_{1i}^\dagger \partial^m c_{2i} + (-1)^{m+1} c_{2i}^\dagger \partial^m c_{1i}, \quad O^{0,-2}_m = c_{2i}^\dagger \partial^m c_{1i} + (-1)^{m+1} c_{1i}^\dagger \partial^m c_{2i}. \tag{3.2.5}$$

For $(0, 0)$ operators the scaling dimensions are determined by the following matrix:

$$K_{(0,0)} = \frac{1}{1 + 8\alpha^2} \begin{pmatrix} \frac{2}{3}K_c - \frac{1}{3}K_T^c + \frac{8\alpha^2}{3}K_c & \frac{8\alpha^2}{3}(K_c - K_T^c) \\ \frac{8\alpha^2}{3}(K_c - K_T^c) & \frac{2}{3}K_c - \frac{1}{3}K_T^c + \frac{8\alpha^2}{3}K_c \end{pmatrix}, \tag{3.2.6}$$

where we define $K_c$ as the conformal kernel of a single SYK/tensor model with Majorana fermions:

$$K_c(\tau_1, \tau_2, \tau_3, \tau_4) = -\frac{3}{4\pi} \frac{\text{sgn}(\tau_{13})\text{sgn}(\tau_{24})}{|\tau_{13}|^{2\Delta}|\tau_{24}|^{2\Delta}|\tau_{34}|^{2-4\Delta}}, \quad \Delta = \frac{1}{4}, \tag{3.2.7}$$

which has eigenvalues in the anti-symmetric and symmetric sectors as $g_a(h), 3g_s(h)$, with

$$g_a(h) = -\frac{3}{2} \frac{\tan\left(\frac{\pi}{2}(h - \frac{1}{2})\right)}{h - \frac{1}{2}}, \quad g_s(h) = -\frac{1}{2} \frac{\tan\left(\frac{\pi}{2}(h + \frac{1}{2})\right)}{h - \frac{1}{2}}, \tag{3.2.8}$$

and $K_T^c(\tau_1, \tau_2, \tau_3, \tau_4) = K_c(\tau_1, \tau_2, \tau_4, \tau_3)$.

For the $(0, \pm 2)$ operators, they have the same anomalous dimensions determined by

$$K_{(0,\pm 2)} = \frac{8\alpha K_c - 8\alpha^2 K_T^c}{3(1 + 8\alpha^2)}. \tag{3.2.9}$$

As a result, the scaling dimensions of the bilinear operators $\{O^{0,0}_m, O^{0,0}_m, O^{0,2}_m, O^{0,-2}_m\}$ are
determined by equating to 1 the following functions:

\[
\begin{align*}
\{ g_a(h), \frac{3 - 8\alpha^2}{3(1 + 8\alpha^2)} g_a(h), \frac{8\alpha(\alpha + 1)}{3(1 + 8\alpha^2)} g_a(h), \frac{8\alpha(\alpha + 1)}{3(1 + 8\alpha^2)} g_a(h) \} \quad & m \text{ odd}, \\
\{ g_s(h), g_s(h), \frac{8\alpha(1 - \alpha)}{1 + 8\alpha^2} g_s(h), \frac{8\alpha(1 - \alpha)}{1 + 8\alpha^2} g_s(h) \} \quad & m \text{ even}.
\end{align*}
\] (3.2.10)

The series of scaling dimensions coming from solving \( g_a(h) = 1 \) and \( g_s(h) = 1 \) are the same as those found in a single complex SYK model or the \( SU(N)^2 \times O(N) \times U(1) \) tensor model [15]. Thus, for any \( \alpha \neq \frac{1}{4} \), there are two \( h = 1 \) modes corresponding to the \( U(1) \times U(1) \) symmetry.

For \( \alpha = \frac{1}{4} \), we find

\[
\{ g_a(h), \frac{5}{9} g_a(h), \frac{5}{9} g_a(h), \frac{5}{9} g_a(h) \}, \quad \{ g_s(h), g_s(h), g_s(h), g_s(h) \}.
\] (3.2.11)

Thus, four modes with \( h = 1 \) are present. They are solutions with the smallest dimensions in their series, and correspond to operators

\[
c_1^\dagger c_{1i} \pm c_2^\dagger c_{2i}, \quad c_1^\dagger c_{2i} \pm c_2^\dagger c_{1i},
\] (3.2.12)

which are proportional to the generators of the \( U(2) \) symmetry \( \frac{1}{2} c_{si}^\dagger \sigma^a_{ss'} c_{s'i} \).

In contrast to the coupled Majorana SYK model [1], the large \( N \) operator spectrum (3.2.10) does not exhibit a duality symmetry. A duality (3.3.7 can be explored at level of DS equations after assuming certain symmetries on the correlators, but fluctuations not obeying such symmetries prevent this duality from being exact. For example, the theory at \( \alpha = 1 \) is not equivalent to that at \( \alpha = 0 \). For \( \alpha = 1 \), we note that the operator \( O^{0,0}_{1,-} = c_{1i}^\dagger \partial_c c_{1i} - c_{2i}^\dagger \partial_c c_{2i} \) has dimension \( h \approx 1.2829 \). Since this lies in the range \( 1 < h < \frac{3}{2} \), the conformal solution might not be described by a Schwarzian theory [21]. In fact, \( O^{0,0}_{1,-} \) has scaling dimension in this range when \( \alpha > \sqrt{\frac{3}{8}} \).

For \( \alpha < 0 \) or \( \alpha > 1 \) the nearly conformal phase becomes unstable because the scaling dimension of operators \( O_0^{0,\pm 2} \) becomes complex. The plot of its imaginary part as a function
of $\alpha$ is in fig. 3.1. We note that it reaches its maximum when $\alpha = -1/2$. The antisymmetric sector cannot have such an instability for any $\alpha$ since $-1/3 < \frac{3-8\alpha^2}{3(1+8\alpha^2)} \leq 1$ and $-1/3 < \frac{8\alpha(\alpha+1)}{3(1+8\alpha^2)} \leq 2/3$. So the lower bound is greater than $1/k_a(1/2) = -4/(3\pi)$. In such cases, the real infrared solution acquires VEV of $O_{0,\pm}^0$ corresponding to the spontaneous breaking of $U(1)_-$ symmetry.

![Figure 3.1: The imaginary part of the scaling dimension of operator $c_{1i}^\dagger c_{2i}$. It reaches its maximum at $\alpha = -1/2$.](image)

### 3.3 General Dyson-Schwinger equations and their numerical solution

In this section we study the DS equations more generally and show that, for $\alpha < 0$ or $\alpha > 1$, the solution with lowest free energy breaks the $U(1)_-$ symmetry. These equations may be obtained by varying the effective action (3.1.8). The first series is

$$
\frac{\partial \tau_1}{\partial \sigma} G_{\sigma \sigma'}(\tau_1, \tau_2) - \int d\tau_3 \Sigma_{\sigma \sigma''}(\tau_1, \tau_3) G_{\sigma'' \sigma'}(\tau_3, \tau_2) = \delta_{\sigma \sigma'} \delta(\tau_{12}).
$$

(3.3.1)
For the second series we find

\[
\Sigma_{11}(\tau_{12}) = -J^2 G_{11}(\tau_{12})^2 G_{11}(\tau_{21}) - 4\alpha J^2 G_{11}(\tau_{12}) \left( G_{12}(\tau_{12}) G_{21}(\tau_{21}) + G_{12}(\tau_{21}) G_{21}(\tau_{12}) \right) \\
- 8\alpha^2 J^2 G_{22}(\tau_{21}) \left( G_{11}(\tau_{12}) G_{22}(\tau_{12}) + G_{12}(\tau_{12}) G_{21}(\tau_{12}) \right),
\]

\[
\Sigma_{12}(\tau_{12}) = -J^2 G_{12}(\tau_{12})^2 G_{21}(\tau_{21}) - 4\alpha J^2 G_{12}(\tau_{12}) \left( G_{11}(\tau_{12}) G_{11}(\tau_{21}) + G_{22}(\tau_{12}) G_{22}(\tau_{21}) \right) \\
- 8\alpha^2 J^2 G_{12}(\tau_{21}) \left( G_{11}(\tau_{12}) G_{22}(\tau_{12}) + G_{12}(\tau_{12}) G_{21}(\tau_{12}) \right). \tag{3.3.2}
\]

The equation for \(\Sigma_{22}(\tau_{12})\) is obtained from \(\Sigma_{11}(\tau_{12})\) by \(G_{11} \leftrightarrow G_{22}\), and that for \(\Sigma_{21}(\tau_{12})\) is obtained from \(\Sigma_{12}(\tau_{12})\) by \(G_{12} \leftrightarrow G_{21}\). In Appendix 3.9 we show how to derive these equations diagrammatically in both the coupled SYK and tensor models.

![Figure 3.2](image_url)

**Figure 3.2:** Numerical solutions to the DS equations for different values of \(\alpha\) and \(\beta J\), plotted against \(\theta = 2\pi \frac{\tau}{\beta}\). All the values of \(\alpha\) shown lie in the range where \(U(1)\) is spontaneously broken at large \(\beta J\). We note that all correlators exponentially decay at the same rate.

One can see that matrix \(G_{\sigma\sigma'}(\tau)\) is Hermitian \(G^\dagger(\tau) = G(\tau)\), which implies that \(G^*_{11}(\tau) = G_{11}(\tau)\) and \(G^*_{12}(\tau) = G_{21}(\tau)\). The Particle-Hole symmetry implies that \(G_{\sigma\sigma'}(\tau) = -G_{\sigma'\sigma}(-\tau)\),
which leads to

$$G_{12}(-\tau) = -G_{21}(\tau) = -G_{12}^*(\tau) .$$  \hfill (3.3.3)

Assuming also that $G_{22}(\tau) = G_{11}(\tau)$, we find for the DS equations

$$J^{-2}\Sigma_{11}(\tau) = (1 + 8\alpha^2)G_{11}(\tau)^3 + 4\alpha G_{11}(\tau)(G_{12}^2(\tau) + G_{12}^*(\tau) + 2\alpha |G_{12}(\tau)|^2) ,$$

$$J^{-2}\Sigma_{12}(\tau) = G_{12}^3(\tau) + 8\alpha G_{12}(\tau)G_{11}^2(\tau) + 8\alpha^2 G_{12}^*(\tau)(G_{11}^2(\tau) + |G_{12}(\tau)|^2) ,$$  \hfill (3.3.4)

together with $\Sigma_{22}(\tau) = \Sigma_{11}(\tau)$ and $\Sigma_{21}(\tau) = \Sigma_{12}^*(\tau)$. We notice that $G_{11}(\tau)$ is real, $G_{11}^*(\tau) = G_{11}(\tau)$, whereas $G_{12}(\tau)$ can be complex. The first series of DS equations then reads

$$\partial_\tau G_{11}(\tau) - \int d\tau' \left( \Sigma_{11}(\tau - \tau')G_{11}(\tau') + \Sigma_{12}(\tau - \tau')G_{12}^*(\tau') \right) = \delta(\tau) ,$$

$$\partial_\tau G_{12}(\tau) - \int d\tau' \left( \Sigma_{11}(\tau - \tau')G_{12}(\tau') + \Sigma_{12}(\tau - \tau')G_{11}(\tau') \right) = 0 .$$  \hfill (3.3.5)

Now we can look for solutions preserving different kinds of discrete symmetries. If we assume that the solution preserves the $Z_4$ symmetry (3.1.4),

\textsuperscript{e} then we have $G_{12}(\tau) = -G_{21}(\tau)$. Combining this with $G_{12}^*(\tau) = G_{21}(\tau)$, we see that $G_{12}$ is purely imaginary. Using also (3.3.3), we find that $G_{12}(\tau) = G_{12}(-\tau)$. Therefore, similarly to [1], we have to solve for only two functions: an odd real one, $G_{11}(\tau) = G_{22}(\tau)$, and an even imaginary one, $G_{12}(\tau)$. The equations determining these two functions are

$$J^{-2}\Sigma_{11}(\tau) = (1 + 8\alpha^2)G_{11}^3(\tau) + 8\alpha(1 - \alpha)G_{11}(\tau)G_{12}^2(\tau) ,$$

$$J^{-2}\Sigma_{12}(\tau) = (1 + 8\alpha^2)G_{12}^3(\tau) + 8\alpha(1 - \alpha)G_{12}(\tau)G_{11}^2(\tau) .$$  \hfill (3.3.6)

They are very similar to the equations derived in [1]; the functions of $\alpha$ are somewhat

\textsuperscript{e}Alternatively, we may assume an interchange symmetry $c_{1i} \leftrightarrow c_{2i}$, which implies $G_{12}(\tau) = G_{21}(\tau)$. Combining this with $G_{12}^*(\tau) = G_{21}(\tau)$, we see that $G_{12}$ is now purely real and odd. Thus, we have two odd real functions: $G_{11}(\tau)$ and $G_{12}(\tau)$. In this phase there cannot be a VEV of operator $c_{1i}c_{2i}^\dagger$, but there can be a VEV of $c_{1i}\partial_\tau c_{2i}^\dagger$. However, the latter is unlikely to appear dynamically. Therefore, the interchange symmetry does not appear to be realized.
different, but they again demonstrate changes of behavior at \( \alpha = 0 \) and 1. The solutions to these equations may be obtained similarly to those in [1], and they are plotted in fig. 3.2.

We note that there is a duality symmetry of (3.3.6): these equations are invariant under

\[
J \rightarrow \frac{1 + 8\alpha}{3} J , \quad \alpha \rightarrow \frac{1 - \alpha}{1 + 8\alpha} .
\]

(3.3.7)

However, this is not a symmetry of the theory even in the large \( N \) limit: neither (3.3.4), nor the bilinear spectrum (3.2.10) respect it.

Due to the underlying \( U(1) \) symmetry, there is a continuous family of solutions obtained from these ones through the transformation \( G_{12}(\tau) \rightarrow e^{i\phi} G_{12}(\tau) \). If we don’t a priori assume the \( Z_4 \) symmetry (3.1.4), we find that the general numerical algorithm typically converges to a solution of this form with some phase \( \phi \). We note that such a solution has a modified discrete symmetry \( c_{1j} \rightarrow e^{-i\phi} c_{2j}, \ c_{2j} \rightarrow -e^{i\phi} c_{1j} \).

Let us calculate the expectation values of the \( U(1) \times U(1) \) charges. After introducing a point splitting regulator and writing

\[
Q_1 = \lim_{\epsilon \to 0} \frac{1}{2} [c_{1i}^\dagger(\epsilon), c_{1i}(0)] ,
\]

(3.3.8)

it follows that

\[
\langle Q_1 \rangle = \frac{1}{2} \lim_{\epsilon \to 0^+} (G_{11}(\epsilon) + G_{11}(-\epsilon)) = \frac{1}{2} \lim_{\epsilon \to 0^+} (G_{11}(\epsilon) - G_{11}(\beta - \epsilon)) .
\]

(3.3.9)

Since for the solution in fig.3.2 \( G_{11}(\tau) = G_{22}(\tau) \) has the symmetry \( G_{11}(\tau) = G_{11}(\beta - \tau) \), we see that

\[
\langle Q_+ \rangle = \lim_{\epsilon \to 0} \frac{1}{2} (G_{11}(\epsilon) - G_{11}(\beta - \epsilon) + G_{22}(\epsilon) - G_{22}(\beta - \epsilon)) = 0 .
\]

(3.3.10)

Analogously, we see that \( \langle Q_- \rangle = 0 \). Since \( U(1)_+ \) is unbroken, \( \langle Q_+ \rangle = 0 \) indicates that
any ground state must have $Q_+ = 0$. For $U(1)_-$, a ground state admits decomposition $|0\rangle = \sum_n c_n |n\rangle$, and $\langle Q_- \rangle = 0$ simply follows from the charge conjugation symmetry we imposed in our solution.

$$|0\rangle = \sum_n c_n |n\rangle,$$

Figure 3.3: The gap $\Delta E$ (in units where $J = 1$) for negative $\alpha$ calculated from the exponential decay of the solutions to the DS equations. We obtain the gap by linear fitting $\log |G_{11}|$ in regime $\frac{1}{J} \ll \tau \ll \beta$, where the solution is dominated by the exponential decay, and the exponent is dominated by $\Delta E$ at zero temperature.

The exponential decay in fig. 3.2 indicates an $\mathcal{O}(1)$ gap at large $N$ between the ground state, which has $Q_+ = 0$, and the state with the lowest energy in the $Q_+ = 1$ sector, i.e.

$$\Delta E = E_0(Q_+ = 1) - E_0(Q_+ = 0) . \quad (3.3.11)$$

To see this, consider inserting a complete set of states

$$\langle c^\dagger_{\sigma}(\tau)c_{\sigma'}(0) \rangle = \sum_n e^{(E_0 - E_n)\tau} \langle 0|c^\dagger_{\sigma}|n\rangle \langle n|c_{\sigma'}|0\rangle , \quad (3.3.12)$$

where $\sigma, \sigma'$ ranges from 1 to 2. In order for the matrix elements to be non-vanishing, $|n\rangle$ must have $Q_+ = 1$. Using the numerical solutions to DS equations, extrapolated to large $\beta J$, we have plotted in fig. 3.3 the quantity $\Delta E$ from (3.3.11).
Figure 3.4: The magnitude of the off diagonal correlator at $\tau = 0$, corresponding to the VEV of the operator $c_1 c_2^\dagger$, plotted as a function of $\beta$ (we use units where $J = 1$). A similar symmetry breaking behavior is observed for other values $\alpha < 0$ or $\alpha > 1$.

Figure 3.5: Numerical calculation of the large $N$ free energy and entropy at $\alpha = -1/2$. Similarly to [1], for fixed $\alpha$ we observe a second-order phase transition from the $U(1)$ symmetric phase to $U(1)$ broken phase. We numerically observe that $S/N$ approaches zero, rather than a finite number, as $\beta \to \infty$. This may be explained by the $U(1)$ sigma model, where one expects $S_0 \sim \log N$ instead of powers in $N$.

Given the DS solution, we can also calculate the ground state energy via

$$
\langle 0 | H | 0 \rangle = \lim_{\epsilon \to 0^+} \frac{1}{2} \left( \langle c_{1i}^\dagger (\tau + \epsilon) \partial_\tau c_{1i}(\tau) \rangle + \langle c_{2i}^\dagger (\tau + \epsilon) \partial_\tau c_{2i}(\tau) \rangle \right) = \lim_{\tau \to 0^+} \partial_\tau G(\tau). \quad (3.3.13)
$$
In momentum space this is given by
\[ E_0 = \frac{1}{\beta} \sum_n (\Sigma_{11}(\omega_n) G_{11}(\omega_n) - \Sigma_{12}(\omega_n) G_{12}(\omega_n)) . \] (3.3.14)

We find good agreement between the DS computation of the ground state energy and the exact diagonalization results, as summarized in fig. 3.11.

### 3.4 Charge compressibility and the sigma model

Since the \( U(1) \) symmetry is spontaneously broken for \( \alpha < 0 \) and \( \alpha > 1 \), we expect the presence of a gapless Goldstone mode. It arises from the degeneracy between ground states in sectors with different values of the charge \( Q_- \), which emerges in the large \( N \) limit. The expected action for the Goldstone modes is the \( U(1) \) sigma model action:
\[ S_{U(1)_-} = \frac{N K_-}{2} \int d\tau (\partial_\tau \phi(\tau))^2, \quad \phi \sim \phi + 2\pi, \] (3.4.1)

where the coefficient \( K_- \), which is \( \mathcal{O}(1) \) in the large \( N \) limit, is the zero-temperature compressibility for the \( U(1)_- \) charge.

Let us emphasize that this \( U(1)_- \) sigma model has a completely different origin from the \( U(1) \) sigma model arising in the complex SYK model, which was recently discussed in detail in [39]. In the latter case, the physics is similar to the conventional SYK model: there is an approximate conformal symmetry in the IR, the Schwartzian effective action, zero-temperature entropy and, most importantly, \( U(1) \) symmetry is not broken. The sigma model in this case has the same origin as the Schwartzian action, since dropping the fermionic kinetic term promotes the global \( U(1) \) symmetry to local \( U(1) \). The finite \( 1/J \) corrections manifest themselves in the time-reparametrization Schwartzian mode and the \( U(1) \)-phase reparametrization sigma-model.

Assuming that in the range \( 0 < \alpha < 1 \) the solution is given by the standard near-
conformal SYK saddle, so there are no anomalous VEVs, we essentially have two non-interacting complex fermions. In particular, when chemical potential $\mu_+ \neq 0$, the system has solution $G_{11}(\tau) = G_{22}(\tau)$, and when $\mu_- \neq 0$, $G_{11}(\tau) = G_{22}(\beta - \tau)$. Both reduce (3.3.1) and (3.3.2) to that of two decoupled complex fermions with chemical potentials $\mu_{\pm}$. We then find that we have two sigma-models, for $U(1)_{\pm}$, with compressibilities:

$$K_- = K_+ = \frac{2K_{c\text{SYK}}}{\sqrt{1 + 8\alpha^2}},$$

(3.4.2)

where $K_{c\text{SYK}} \approx 1.04$ is the compressibility of a single complex SYK model [39]. The factor of two comes from having two fermions, and the square root comes from renormalization of $J$ by non-zero $\alpha$, (3.2.2). Let us point out though that, at $\alpha = 1/4$, the $U(1)_-$ symmetry is enhanced to $SU(2)$. We will discuss this case separately in section 3.6.

In the case of spontaneously broken $U(1)_-$ symmetry, the physics is different. The solutions of the Dyson-Schwinger equations that we have found for $\alpha < 0$ do not have a conformal form. Therefore, there is no approximate reparametrization symmetry or Schwartzian effective action. In the large $N$ limit, the action (3.4.1) is a conventional Nambu-Goldstone mode action. On these grounds, we do not expect to have a sigma model for $U(1)_+$ symmetry, since it is unbroken. Therefore, the splittings between sectors with different values of $Q_+$ should not vanish in the large $N$ limit. This implies that the compressibility $K_+$ defined as $dQ_+/d\mu_+$ is zero, so that a small chemical potential does not generate non-zero charge. We will see this in the large $N$ DS equations momentarily. In the exact diagonalization at finite $N$, this manifests in the fact that the energy dependence on $Q_+$ is not close to quadratic.

Let us return to the $U(1)_-$ symmetry and compute the corresponding compressibility $K_-$. It can be found in three ways: First of all, it is the derivative of the charge with respect to the chemical potential:

$$K_- = \frac{dQ_-}{d\mu_-}, \quad \text{at } T = 0, \mu_- = 0.$$

(3.4.3)
Secondly, it is related to the grand canonical thermodynamical potential \( \Omega \) as

\[
\Omega = \Omega_0 - \frac{NK_- \mu_-^2}{2}, \quad T = 0, \quad (3.4.4)
\]

and finally the action (3.4.1) can be quantized leading to the spectrum:

\[
E_{Q_-} = A + \frac{Q_-^2}{2NK_-}, \quad Q_- \in \mathbb{Z}. \quad (3.4.5)
\]

Let us emphasize that the \( U(1)_- \) symmetry breaking occurs only in the limit \( N \to \infty \). In systems with finite numbers of degrees of freedom this does not happen. From the above spectrum we see how it happens: if \( N = \infty \) we have a classical particle on a circle (3.4.1) with an infinite number of classical vacua. However, finite \( N \) effects quantize the action, leading to a unique ground state and spectrum (3.4.5).

It will be convenient for us to find \( K_- \) numerically by introducing a chemical potential into the large \( N \) Dyson-Schwinger equations and fitting the numerical result for \( \Omega \) using eq. (3.4.4). In fact, to double check our results, we will introduce chemical potentials \( \mu_- \) and \( \mu_+ \) for \( U(1)_- \) and \( U(1)_+ \) and fit \( \Omega \) with

\[
\Omega = \Omega_0 - \frac{NK_- \mu_-^2}{2} - \frac{NK_+ \mu_+^2}{2} - NK_{\text{mix}} \mu_- \mu_+. \quad (3.4.6)
\]

Since \( U(1)_+ \) is unbroken, we expect that \( K_+ = K_{\text{mix}} = 0 \). In other words, the low energy states are not charged under \( U(1)_+ \), and the gap to states with non-vanishing \( U(1)_+ \) charges is big. The result is presented in Figure 3.6. We indeed see that \( K_+ = K_{\text{mix}} = 0 \).

Finally, we illustrate our claims by plotting the Green function \( G_{11} \) upon introducing \( \mu_\pm \) in fig. 3.7. These results were obtained by solving the DS equations numerically with \( J = 1, \beta = 40, \alpha = -1.5 \) and \( \mu_\pm = 0.3 \). The charge \( Q_+ \) remains zero despite the fact that the Green functions are no longer symmetric, whereas \( Q_- \) is definitely non-zero.
Figure 3.6: The numerical results for three different compressibilities as functions of $\alpha$ for $J = 1$, $\beta = 100$. We checked that the result does not depend appreciably on $\beta$ by comparing with the $\beta = 50$ data.

Figure 3.7: Left: $Q_+$ is zero even when $\mu_+ \neq 0$. Right: $Q_-$ is generated for $\mu_- \neq 0$.

3.5 Results from Exact Diagonalizations

In this section we will study the energy spectra for accessible values of $N$. We will use the particle-hole symmetric version of the Hamiltonian, given in (3.7.4). We have generated multiple random samples of the Hamiltonain, which allow us to study various averaged quantities as functions of $\alpha$ and $N$.

3.5.1 Evidence for Symmetry Breaking

For $\alpha < 0$ and $\alpha > 1$, the large $N$ DS equations indicate that $U(1)_-$ symmetry is spontaneously broken. In these ranges of $\alpha$, the absolute ground state appears in the sectors with $Q_+ = 0$ and the lowest possible value of $|Q_-|$, which is $|Q_-| = 0$ for even $N$ and $|Q_-| = 1$
for odd $N$. This means that, for odd $N$, there are two degenerate ground states, which have $Q_- = \pm 1$, and their mixture admits an expectation value of operator $c_1^\dagger c_{2i}$ already at finite $N$. At any finite even $N$ we cannot see the spontaneous symmetry breaking, but it appears in the large $N$ limit due to the degeneracy of ground states with $Q_+ = 0$ and different values of $Q_-$. 

![Density of states in two of the charge sectors](image)

Figure 3.8: Density of states in two of the charge sectors, $(Q_+, Q_-) = (0, 0)$ and $(2, 0)$, for a single realization of the model with $N = 10$ and $\alpha = -1/2$. The lower plots are zoomed in regions near the ground state. We observe a prominent gap in the $(Q_+, Q_-) = (0, 0)$ sector.

In fig. 3.8 we exhibit the spectra in two different charge sectors for $N = 10$. A characteristic quantity in the broken symmetry phase is the gap between the first excited state and the ground state: such a gap is observed in the sectors with $Q_+ = 0$. For example, for the special value $\alpha = -1/2$ some of the charge sectors contain a large number of states with exactly zero energy, and this number is independent of the sampling of $J_{ij,kl}$. An analogous phenomenon was observed in [1] for the coupled Majorana model with $\alpha = -1$. 

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1For the special value $\alpha = -1/2$ some of the charge sectors contain a large number of states with exactly zero energy, and this number is independent of the sampling of $J_{ij,kl}$. An analogous phenomenon was observed in [1] for the coupled Majorana model with $\alpha = -1$. 

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in the \((Q_+, Q_-) = (0, 1)\) sectors we find for \(\alpha = -1/2\) that the average gaps above the ground state are \(\approx 0.440, 0.437, 0.473\) for \(N = 7, 9, 11\), respectively. These results suggest that the gap is non-vanishing in the large \(N\) limit.

Similarly, there is a sizable difference between the ground state energies in sectors with different values of \(Q_+\). It is noticeably bigger than the difference between sectors with different values of \(Q_-\), which is expected to be of order \(1/N\). For example, for \(\alpha = -1/2\) and \(N = 10\), we find

\[
E_0(Q_+ = 2, Q_- = 0) - E_0(Q_+ = 0, Q_- = 0) \approx 0.622 ,
\]
\[
E_0(Q_+ = 0, Q_- = 2) - E_0(Q_+ = 0, Q_- = 0) \approx 0.236 .
\] (3.5.1)

In the sectors with \(Q_+ = 0\), we expect the ground state energies to depend quadratically on \(Q_-\):

\[
E_0 = A(\alpha, N) + \frac{Q_-^2}{2B_-(\alpha, N)} ,
\]
\[
B_-(\alpha, N) = K_-(\alpha)N + C_-(\alpha) + O(1/N) ,
\] (3.5.2)

where \(K_-\) is the large \(N\) compressibility for the \(U(1)_-\) degree of freedom. As can be seen in figs. 3.9 and 3.10, these quadratic fits work well, and \(B_-(\alpha)\) is approximately linear in \(N\). From the slopes we find that \(K_-^{ED}(-0.5) \approx 0.87\) and \(K_-^{ED}(2) \approx 0.33\). These values of compressibility are close to those obtained from the Dyson-Schwinger calculations directly in the large \(N\) limit: \(K_-^{DS}(-0.5) \approx 0.80\) and \(K_-^{DS}(2) \approx 0.32\).

Another important quantity is the leading term in the ground state energy (3.5.2), \(A(\alpha, N)\), which is expected to grow linearly for large \(N\). In fig. 3.11 we plot \(A(\alpha, N)\) for \(\alpha = -0.5\) and \(\alpha = -0.2\), and show the fits

\[
A(\alpha, N) = \mathcal{E}_0(\alpha)N + D(\alpha) + O(1/N) .
\] (3.5.3)

In fig, 3.12 we plot \(\mathcal{E}_0(\alpha) = \lim_{N \to \infty} E_0(\alpha)/N\) for a range of negative \(\alpha\). This shows good
Figure 3.9: The $E(Q)$ curve (left) at $\alpha = -1/2$ for the first few $Q_-$ sectors at $Q_+ = 0$. For each $E(Q)$ curve, we make a quadratic fit $E_0 = \frac{Q^2}{2B_-} + A$ and determine $K_-$ from the linear fit of $B_-$ vs. $N$. The slope of the plot for $\alpha = -1/2$ is $K_{\text{ED}}^\approx 0.87$, which agrees well with the DS calculation of $K_{\text{DS}}^\approx 0.80$.

Figure 3.10: The $E(Q)$ curve (left) at $\alpha = 2$ for the first few $Q_-$ sectors at $Q_+ = 0$. The slope of the plot of $B_-$ vs. $N$ is $K_{\text{ED}}^\approx 0.33$, which is not far from the DS calculation of $K_{\text{DS}}^\approx 0.32$.

agreement with the corresponding calculation using DS equation as a function of $\alpha$.

3.5.2 Line of Fixed Points

Along the fixed line $0 \leq \alpha \leq 1$ there is no symmetry breaking, and the large $N$ spectrum is gapless in every charge sector. In fact, for such values of $\alpha$, near the edge the density of
Figure 3.11: Plots of the leading term in the ground state energy (3.5.2), \( A(\alpha, N) \), vs. \( N \) for \( \alpha = -0.5 \) (left) and \( \alpha = -0.2 \) (right). The linear fits determining the slope, \( \mathcal{E}_0(\alpha) \), are also shown.

Figure 3.12: Comparison of the ED and DS calculations of \( \mathcal{E}_0(\alpha) = \lim_{N \to \infty} E_0(\alpha)/N \). They show good agreement even though the ED results are available only up to a moderate values of \( N \).

The state should behave as

\[
\rho_{2c\text{SYK}}(E) = \int dE' \rho_{c\text{SYK}}(E - E') \rho_{c\text{SYK}}(E') \sim E^2.
\]  

(3.5.4)

Along the fixed line, we expect the gaps to be of order \( 1/N \) for excitations of both the
\( Q_- \) and \( Q_+ \) charges, so that both \( U(1)_- \) and \( U(1)_+ \) compressibilities are well-defined:

\[
E_0 = A(\alpha) + \frac{Q_-^2}{2B_-(\alpha, N)} + \frac{Q_+^2}{2B_+(\alpha, N)},
\]

\[
B_{\pm}(\alpha, N) = K_{\pm}(\alpha)N + C_{\pm}(\alpha) + O(1/N). \tag{3.5.5}
\]

For \( 0 < \alpha < 1 \), we find that \( B_+ > B_- \) for all the values of \( N \) we have studied. This leads to the fact that, for odd \( N \), the ground state does not have \( Q_+ = 0 \). Indeed, for odd \( N \), the lowest possible values of \((Q_+, Q_-)\) are \((0, \pm 1)\) and \((\pm 1, 0)\). Since \( B_+ > B_- \), there are two ground states with \( Q_+ = \pm 1, Q_- = 0 \) for odd \( N \). On the other hand, for even \( N \) there is a unique ground state with \( Q_+ = Q_- = 0 \).

For \( \alpha = 0 \), \( B_- = B_+ \). Therefore, the compressibilities are equal: \( K_+(0) \approx K_-(0) \approx 2.08 \). Indeed, for \( \alpha = 0 \) the Hamiltonian is simply a sum of two cSYK Hamiltonian with the common \( J_{ijkl} \), so that for large \( N \)

\[
E_0 = A(0) + \frac{1}{2NK_{cSYK}} \left( Q_1^2 + Q_2^2 \right). \tag{3.5.6}
\]

To compare with our normalizations, \( K_{cSYK} = K_+(0)/2 \approx 1.04 \). Thus, our finding for \( \alpha = 0 \) is in good agreement with the result \( K_{cSYK} \approx 1 \) from [39].

We have also done fits of the two large \( N \) compressibilities along the fixed line. Then \( K_- \) is found to be smaller than \( K_+ \). This is in conflict with the DS calculations giving equal values, which may be due to the slow convergence of the ED results to the large \( N \) limit. The DS formula \( K_+(\alpha) = K_+(0)/\sqrt{1 + 8\alpha^2} \) predicts the value \( \approx 1.96 \) at \( \alpha = 0.125 \), and \( \approx 0.94 \) at \( \alpha = 0.7 \). If we a priori assume \( K_- = K_+ = K(ED) \) in the ED fit for \( \alpha > 0 \), we obtain results in quite good agreement with the DS calculations. For example, at \( \alpha = \frac{1}{8} \), \( K(ED) \approx 1.92 \) vs \( K(DS) \approx 1.96 \). At \( \alpha = 0.7 \), we get \( K(ED) \approx 0.95 \) vs \( K(DS) \approx 0.94 \).
3.6 The $U(2)$ symmetric model

A special case is $\alpha = 1/4$ where the Hamiltonian becomes (3.0.2), and the symmetry is enhanced to $SU(2) \times U(1)_+$. In this section we assemble various results at this special point, which is interesting because it corresponds to an SYK-like model with a non-abelian global symmetry [87–90].

Due to the $SU(2)$ symmetry, there are some exact degeneracies in the spectrum between states with different values of $Q_- = 2S_z$. The states naturally split into sectors labeled by the $U(1)_+$ charge $Q_+$ and the $SU(2)$ spin $S$. In fig. 3.13 we show the histogram for the $U(2)$ invariant states, which have $Q_+ = S = 0$. Such states appear only when $N$ is even, and the unique absolute ground state is in this sector. The histogram was obtained from a single realization of the Hamiltonian (3.7.4) with $N = 10$, and it shows that the $U(2)$ symmetric theory is in the gapless phase.

![Figure 3.13: Density of states in the $(Q_+, S) = (0, 0)$ sector for a single realization at the $U(2)$ symmetric point $\alpha = 1/4$ for $N = 10$. The ground state is in this $U(2)$ invariant sector. On the right we enlarge the region near the ground state; this shows that there is no significant gap.](image)

Let us discuss the low-energy effective action for the $U(2)$ symmetric theory. We expect that instead, of the $U(1)_+ \times U(1)_- \sigma$ model, we now have $SU(2) \times U(1)_+$. The low
energy effective action for the $SU(2)$ part is:

$$S_{SU(2)} = - \frac{B_{SU(2)}}{4} \int d\tau \text{Tr} \left(U^\dagger \partial_\tau U\right)^2,$$

$$B_{SU(2)} = NK_{SU(2)} + C_{SU(2)} + O(1/N), \quad (3.6.1)$$

where $U(\tau)$ is a $SU(2)$ matrix variable. Previously, we obtained compressibilities from coupling to $\mu_-$ chemical potential. Let us argue that this calculation does not change. Indeed, corrections $\propto N$ to the free energy depend on classical properties of this sigma-model, since we have a factor of $N$ in front. Upon introducing a chemical potential for the $U(1)_-$ subgroup of $SU(2)$, we have to study the following action:

$$- \frac{B_{SU(2)}}{4} \int d\tau \text{Tr} \left(U^\dagger \partial_\tau U + \text{diag}(\mu_-, -\mu_-)\right)^2. \quad (3.6.2)$$

Its contribution to the Gibbs potential is again $-B_{SU(2)}\mu_-^2/2$. We find using the DS equations that

$$K_{SU(2)} \approx 1.7. \quad (3.6.3)$$

However, the low energy spectrum is very different, as it involves quantizing the sigma model. Namely, now the excitations come in $SU(2)$ multiplets with energies given by a quadratic Casimir of $SU(2)$. Namely, for a multiplet with $Q_-/2 \in (-S, -S + 1, \ldots, S)$ the energy is given by:

$$\delta E = \frac{2S(S + 1)}{NK_{SU(2)} + C_{SU(2)}}. \quad (3.6.4)$$

Therefore, we find for large $N$:

$$E_0 \approx A + \frac{1}{2(NK_+ + C_+)}Q_+^2 + \frac{2}{NK_{SU(2)} + C_{SU(2)}}S(S + 1), \quad (3.6.5)$$

where $S$ is the $SU(2)$ spin. For even $N$, the unique ground state occurs in the $Q_+ = S = 0$. For odd $N$, there are two ground states: they are $SU(2)$ singlets and have $Q_+ = \pm 1$. 

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A priori, there are two different compressibilities; see fig. 3.14. Fitting them separately, we find $K_{SU(2)} \approx 1.45$ and $K_+ \approx 1.78$. The fact that they are different disagrees with the DS results; this could be due to the fact that our data does not access large enough $N$. However, if we assume that they are equal, then the fit value is $K_+ = K_{SU(2)} \approx 1.6$, which is not far from the DS value (3.6.3).

Figure 3.14: The dependence of ground state energy at $\alpha = 0.25$ on $SU(2)$ spin $S$ at $Q_+ = 0, 1, 2$. We use the Ansatz $E_0 \approx A + \frac{1}{2B_+}Q^2 + \frac{2}{B_{SU(2)}S(S+1)}$, and plot $B_{SU(2)}$ against $N$ to estimate the compressibility, $K_{SU(2)} \approx 1.45$. 
3.7 Appendix: Particle-hole symmetry

For the single complex SYK model, the Hamiltonian which respects the particle-hole symmetry $c_i \leftrightarrow c_i^\dagger$, accompanied by $J_{ijkl} \rightarrow J_{ijkl}^*$, was given in [39]

$$H_{c\text{SYK}} = \sum_{i,j,k,l=1}^{N} J_{ijkl} A\{ c_i^\dagger c_j^\dagger c_k c_l \}, \quad (3.7.1)$$

where $A$ denotes total antisymmetrization:

$$A\{ c_i^\dagger c_j^\dagger c_k c_l \} = c_i^\dagger c_j^\dagger c_k c_l + \frac{1}{2} \left( \delta_{ik} c_j^\dagger c_l - \delta_{il} c_j^\dagger c_k + \delta_{jl} c_i^\dagger c_k - \delta_{jk} c_i^\dagger c_l + \frac{1}{2} (\delta_{il} \delta_{jk} - \delta_{ik} \delta_{jl}) \right). \quad (3.7.2)$$

To make the Hamiltonian of the coupled model, (3.0.1), invariant under the full particle-hole symmetry (3.1.5), we have to add to it similar terms:

$$H_{ed} = \sum_{i,j,k,l=1}^{N} J_{ijkl} \left( A\{ c_{i1}^\dagger c_{j1}^\dagger c_{1k} c_{1l} \} + A\{ c_{i2}^\dagger c_{j2}^\dagger c_{2k} c_{2l} \} ight)$$

$$+ 8\alpha \left( c_{i1}^\dagger c_{j2}^\dagger c_{2k} c_{1l} - \frac{1}{2} \delta_{jk} c_{i1}^\dagger c_{1l} - \frac{1}{2} \delta_{ik} c_{j2}^\dagger c_{2l} + \frac{1}{4} (\delta_{il} \delta_{jk}) \right). \quad (3.7.3)$$

This can also be written as

$$H_{ed} = \sum_{i,j,k,l=1}^{N} J_{ijkl} \left( c_{i1}^\dagger c_{j1}^\dagger c_{1k} c_{1l} + c_{i2}^\dagger c_{j2}^\dagger c_{2k} c_{2l} + 8\alpha c_{i1}^\dagger c_{j2}^\dagger c_{2k} c_{1l} + (1 + 2\alpha) \left( -2\delta_{jk} c_{i1}^\dagger c_{1l} - 2\delta_{il} c_{j2}^\dagger c_{2k} + \delta_{il} \delta_{jk} \right) \right). \quad (3.7.4)$$

The quadratic and c-number terms are subleading in $N$ and thus are not important at large $N$. They can be important at small $N$, such as in the exact diagonalizations. We note that these terms vanish for $\alpha = -1/2$, so that the original Hamiltonian (3.0.1) is automatically particle-hole symmetric for this value of $\alpha$. At another special value, $\alpha = 1/4$, the Hamiltonian 3.7.4 respects the $U(2)$ symmetry possessed by the purely quartic
3.8 Appendix: Zero modes of the quadratic fluctuations

In this section we give an alternative derivation of the scaling dimension of various primary operators by looking at zero modes of the quadratic fluctuations near the nearly conformal saddle points of the effective action 3.1.8. We assume the time translational invariance and study fluctuations around the symmetric saddle point. The zero modes of the quadratic fluctuation correspond to the operator three point functions \( \delta G_{\sigma \sigma'}(\tau) = \langle \frac{1}{N} c_{\sigma i}^\dagger(\tau) c_{\sigma' i}(0) \mathcal{O}_h(\infty) \rangle \), because the DS equations hold up to arbitrary insertion as long as operators are not inserted at \( \tau \) or 0. In order to not add more contact terms, the operator has to be inserted at \( \infty \). Therefore \( \delta G_{\sigma \sigma'}(\tau) \) would correspond to a zero mode in the quadratic fluctuation, and the eigenvector dictates the form of the operator. Note in conformal theory, the 3 point functions between primaries are determined up to a constant

\[
v(\tau) = \langle \frac{1}{N} c_{\sigma i}^\dagger(\tau) c_{\sigma' i}(0) \mathcal{O}_h(\infty) \rangle = \frac{c \text{sgn}(\tau)}{|\tau|^{2\Delta - h}}, \tag{3.8.1}
\]

where \( h \) is the scaling dimension of the operator \( \mathcal{O} \). In order for the three point function to be non-vanishing, the primary operator \( \mathcal{O} \) is necessarily bilinear in the elementary fermions, and a \( O(N) \) singlet. Therefore one can use this Ansatz to determine the bilinear operator dimension from the quadratic fluctuation. In the following we are going to omit the integrals over \( \tau_1, \tau_2 \) for brevity.

We are looking for quadratic fluctuations above the conformal saddle point \( G_{s12} = G_{s21} = 0 \) and \( G_{s11} = G_{s22} = G_s \), where \( G_s(-\tau) = -G_s(\tau) \) and satisfies the Schwinger-Dyson
\[ \Sigma_s(\tau) = J^2(1 + 8\alpha^2)G_s^3(\tau), \quad G_s(i\omega_n)(-i\omega_n - \Sigma_s(i\omega_n)) = 1. \quad (3.8.2) \]

We find for the second variation

\[ \delta^2 I = \frac{1}{2}G_s(\tau_{41})G_s(\tau_{23})\text{Tr}(\delta \Sigma(\tau_{12})\delta \Sigma(\tau_{34})) - \text{Tr}(\delta \Sigma(\tau_{12})\delta G_{\tau_{21}})) - \frac{J^2}{4}\delta^2 V(G_{ab}). \quad (3.8.3) \]

It will be convenient to introduce two vectors

\[ \delta G(\tau_{12}) = (\delta G_{11}, \delta G_{22}, \delta G_{12}, \delta G_{21}), \quad \delta \Sigma(\tau_{12}) = (\delta \Sigma_{11}, \delta \Sigma_{22}, \delta \Sigma_{21}, \delta \Sigma_{12}) \quad (3.8.4) \]

then we find

\[ \frac{1}{2}G_s(\tau_{41})G_s(\tau_{23})\text{Tr}(\delta \Sigma(\tau_{12})\delta \Sigma(\tau_{34})) = \frac{1}{2}\delta \Sigma^T(\tau_{12})G_s(\tau_{41})G_s(\tau_{23})M\delta \Sigma(\tau_{34}). \]

\[ M = \text{diag}(1, \sigma_x) \]

\[ \delta^2 V = \delta G^T(\tau_{12})G_s^2(\tau_{12})V\delta G(\tau_{34}), \]

\[ V = 2\delta(\tau_{13})\delta(\tau_{24})\text{diag}(1 + 8\alpha^2\sigma_x, 8\alpha^2\sigma_x) - 4\delta(\tau_{14})\delta(\tau_{23})\text{diag}((1 + 4\alpha^2)1 + 4\alpha^2\sigma_x, 4\alpha\sigma_x) \quad (3.8.5) \]

where we used that \( G_s(-\tau) = -G_s(\tau) \). Now we can integrate out fluctuations of \( \delta \Sigma \) fields and find

\[ \delta^2 I = -\frac{1}{2}\delta G^T(\tau_{12})\left((G_s(\tau_{32})G_s(\tau_{14}))^{-1}M + \frac{1}{2}J^2G_s^2(\tau_{12})V\right)\delta G(\tau_{34}). \quad (3.8.6) \]

Now let us introduce new variables \( g(\tau_{12}) = |G_s(\tau_{12})|S\delta G(\tau_{12}), \) where

\[ S = \frac{1}{\sqrt{2}}\text{diag}(\sigma_x - \sigma_z, \sigma_x - \sigma_z), \quad S^T S = 1. \quad (3.8.7) \]
In terms of the new variables, the variation corresponds to operators \( \{ O^m_{1,2}, O^m_{3,4} \} \), where

\[
O^m_{1,2} = c^\dagger_{1i} \partial_t^m c_{1i} + c^\dagger_{2i} \partial_t^m c_{2i} , \quad O^m_{3,4} = c^\dagger_{1i} \partial_t^m c_{2i} + c^\dagger_{2i} \partial_t^m c_{1i} .
\]

We can further decompose \( g \) into symmetric \( g_s(\tau_{12}) = g_s(\tau_{21}) \) and anti-symmetric \( g_a(\tau_{12}) = -g_a(\tau_{21}) \) sectors under time reflection.

Using the new variables we find

\[
\delta^2 I = \frac{3 J^2(1 + 8\alpha^2)}{2} g^T_a(\tau_{12}) \left( K_a^{-1} \text{diag}(1, 1, -1, 1) - \text{diag}(\frac{3 - 8\alpha^2}{3(1 + 8\alpha^2)}, 1, -\frac{8\alpha(\alpha + 1)}{3(1 + 8\alpha^2)}, \frac{8\alpha(\alpha + 1)}{3(1 + 8\alpha^2)}) \right) g_a(\tau_{34})
\[
- \frac{J^2(1 + 8\alpha^2)}{2} g^T_s(\tau_{12}) \left( K_s^{-1} \text{diag}(1, 1, -1, 1) - \text{diag}(1, \frac{8\alpha(\alpha - 1)}{1 + 8\alpha^2}, -\frac{8\alpha(\alpha - 1)}{1 + 8\alpha^2}) \right) g_s(\tau_{34}).
\]

where \( K_a \) and \( K_s \) are standard SYK kernels

\[
K_a(\tau_1, \tau_2; \tau_3, \tau_4) = -3 J^2(1 + 8\alpha^2) |G_s(\tau_{12})| |G_s(\tau_{13})| |G_s(\tau_{24})| |G_s(\tau_{34})|, \\
K_s(\tau_1, \tau_2; \tau_3, \tau_4) = -J^2(1 + 8\alpha^2) |G_s(\tau_{12})| |G_s(\tau_{13})| |G_s(\tau_{24})| |G_s(\tau_{34})| .
\]

The scaling dimensions of the bilinear operators \( \{ O^m_1, O^m_2, O^m_3, O^m_4 \} \) are determined by equating to 1 the functions \((3.2.10)\).

**3.9 Appendix: Diagrammatic Derivation of the Dyson-Schwinger Equations**

The tensor model Hamiltonian \((3.1.11)\) has four vertices, which we call \( v_1, v_2, v_3, v_4 \). We write down Dyson-Schwinger equations for all correlators \( G_{\sigma \sigma'}(\tau, \tau') = \frac{1}{N^3} \langle \psi_1^{\dagger \sigma} \psi_2^{\dagger \sigma'}(\tau) \psi_2^{\alpha \sigma}(\tau') \rangle \), allowed by \( SU(N) \times O(N) \times SU(N) \) symmetries. Note \( \langle \psi_1^{\dagger \sigma} \psi_2^{\dagger \alpha}(t) \psi_2^{\alpha \sigma}(0) \rangle \) is forbidden by \( SU(N) \). This is the tensor counterpart of using the complex \( J_{ij,kl} \) in the SYK model.

At large \( N \), only the melonic diagrams contribute to the leading order in \( N \). The self-
Figure 3.15: The solid line is for $\psi_1$ and dashed $\psi_2$.

Energy can be written in terms of Feynman graphs: Upon drawing the above graphs in the colored line notation, one can check, for example,

$$
\Sigma_{11}(\tau, \tau') = \begin{array}{c}
\quad + \\
\quad + \\
\quad + \\
\quad + \\
\end{array} + \begin{array}{c}
\quad + \\
\quad + \\
\quad + \\
\quad + \\
\end{array} + \begin{array}{c}
\quad + \\
\quad + \\
\quad + \\
\quad + \\
\end{array} + \begin{array}{c}
\quad + \\
\quad + \\
\quad + \\
\quad + \\
\end{array}
$$

$$
\Sigma_{12}(\tau, \tau') = \begin{array}{c}
\quad + \\
\quad + \\
\quad + \\
\quad + \\
\end{array} + \begin{array}{c}
\quad + \\
\quad + \\
\quad + \\
\quad + \\
\end{array} + \begin{array}{c}
\quad + \\
\quad + \\
\quad + \\
\quad + \\
\end{array} + \begin{array}{c}
\quad + \\
\quad + \\
\quad + \\
\quad + \\
\end{array}
$$

colored line notation, one can check, for example,

$$
\Sigma_{11}(\tau) = -g^2 G_{11}(\tau)^2 G_{11}(\tau') - 4g^2 \alpha G_{11}(\tau) G_{12}(\tau) G_{21}(\tau') - 4g^2 \alpha G_{11}(\tau) G_{21}(\tau) G_{12}(\tau') - 2 \times (2\alpha g)^2 G_{11}(\tau) G_{22}(\tau) G_{22}(\tau') - 2 \times (2\alpha g)^2 G_{12}(\tau) G_{21}(\tau) G_{22}(\tau')
$$

$$
= -2 \times (2\alpha g)^2 G_{11}(\tau) G_{22}(\tau) G_{22}(\tau') - 2 \times (2\alpha g)^2 G_{12}(\tau) G_{21}(\tau) G_{22}(\tau')
$$

which exactly agrees with $\Sigma_{11}$ in Eq.(3.3.2). Similarly $\Sigma_{12}$ agrees. To derive the bilinear spectrum, we shall consider ladder diagrams corrections to 3pt functions along the lines of [1], as an effective action at large $N$ is not available for tensor models.
3.10 Appendix: Analytical approximation

If we assume a particular phase such that the $Z_4$ symmetry 3.1.4 is preserved, the DS equations can be written using one function $G = (G_{11} + G_{22} + i (G_{12} - G_{21})) / 2$:

\[-i\omega - \Sigma(\omega))G(-\omega) = -1 ,
\]
\[
\Sigma(\tau) = \tilde{J}^2 \left( G(\tau)^3 + kG(\tau)G(-\tau)^2 \right)
\]

(3.10.1)

where parameter $k$ is related to $\alpha$ by:

\[ k = \frac{3 + 8\alpha + 16\alpha^2}{(4\alpha - 1)^2} , \]

(3.10.2)

and $\tilde{J}$ is related to $J$ in the standard formulation with $G_{11}$ and $\alpha$ by

\[ \tilde{J}^2 = \frac{J^2}{4}(1 - 4\alpha)^2 . \]

(3.10.3)

Let us try to use the following ansatz:

\[ G(\tau) = \begin{cases} 
ae^{-\mu\tau} + \ldots, & \tau > 0 \\
-bc^{-\mu\tau} + \ldots, & \tau < 0 .
\end{cases} \]

(3.10.4)

We have four unknown constants $\mu, c, a, b$, and the dots indicate faster decaying terms. The first DS equation can be rewritten in the time domain as:

\[ \partial_\tau G(\tau) + \int d\tau' \Sigma(\tau' - \tau)G(\tau') = \delta(\tau) . \]

(3.10.5)

Evaluating the convolution for $\tau > 0$ yields:

\[ A_1e^{-\mu\tau} + A_2e^{-(2+c)\mu\tau} + A_3e^{-3c\mu\tau} \]

(3.10.6)
The $A_1, A_{2+c}$ and $A_{3c}$ are easily computed functions of $a, b, c$ and $\mu$:

\[
A_1 = \frac{a^4 c}{4(c + 1)\mu} + \frac{a^4}{4(c + 1)\mu} - \frac{a^3 b k}{(c + 1)\mu} + \frac{a^2 b^2 k}{2(c + 1)\mu} + \frac{ab^3}{\mu - 3c\mu},
\]

(3.10.7)

\[
A_{2+c} = \frac{a^3 b k}{(c + 1)\mu} + \frac{a^2 b^2 k}{2(c + 1)\mu},
\]

(3.10.8)

\[
A_{3c} = \frac{b^4}{4c\mu} - \frac{ab^3}{\mu - 3c\mu}.
\]

(3.10.9)

Terms $e^{-2\mu \tau}, e^{-3c\mu \tau}$ are subdominant and were not present in the ansatz, so we can safely ignore them. Therefore we have a single equation:

\[
A_1 = \frac{a \mu}{J}.
\]

(3.10.10)

For $\tau < 0$ the convolution equals to:

\[
B_c e^{c\mu \tau} + B_{1+2c} e^{(1+2c)\mu \tau} + B_3 e^{3c\mu \tau},
\]

(3.10.11)

where

\[
B_c = \frac{a^3 b}{(c - 3)\mu} + \frac{a^2 b^2 k}{2(c + 1)\mu} - \frac{ab^3 k}{(c + 1)\mu} + \frac{b^4}{4(c + 1)\mu} + \frac{b^4}{4c(c + 1)\mu},
\]

(3.10.12)

\[
B_{1+2c} = \frac{a^2 b^2 k}{2(c + 1)\mu} + \frac{ab^3 k}{(c + 1)\mu},
\]

(3.10.13)

\[
B_3 = \frac{a^4 c}{4(c + 1)\mu} + \frac{a^4}{4(c + 1)\mu} - \frac{a^3 b}{(c - 3)\mu}.
\]

(3.10.14)

Let us assume that $c > 4$. Then we can again ignore the term $e^{(1+2c)\mu \tau}$. However, the term $e^{3c\mu \tau}$ has to be zero. Therefore we have two equations:

\[
B_3 = 0,
\]

(3.10.15)
\[ B_c = \frac{bc\mu}{J}. \]  

(3.10.16)

We see that our ansatz is consistent: we managed to eliminate all faster decaying terms. Moreover, we have 4 unknown variables and only three equations. We will impose one extra condition:

\[ a + b = 1. \]  

(3.10.17)

If ansatz (3.10.4) were an exact solution, then this condition would have followed from having a delta function on the right hand side of DS equations. Unfortunately, (3.10.4) is not an exact solution and at very small \( \tau \) the faster decaying exponential terms become important. However, we still impose eq. (3.10.17) and demonstrate that it agrees with the numerics. So in the end we have four algebraic equations (3.10.10), (3.10.15), (3.10.16), (3.10.17) for four unknown variables \( a, b, c, \mu \). This system can be easily solved numerically.

For comparison, we solve the DS equations numerically for \( \beta \tilde{J} = 400 \) (black dots) and \( \beta \tilde{J} = 1000 \) (red dots) and fixing \( \tilde{J} = 1 \). After that, we fit the numerical solution with exponents (3.10.4). This way we obtain numerical values of \( a, b, c, \mu \). The comparison with analytical answer is presented on Figure 3.16.

Let us note, however, that this approximation does not describe very well the behavior at small Euclidean times \( \tau \). Graphs 3.2 clearly indicate that \( G_{12} \) does not have a linear term near \( \tau = 0 \):

\[ G_{12} = c_1 - c_2 \tau^2, \quad c_1, c_2 > 0. \]  

(3.10.18)

Generically, ansatz (3.10.4) does have a linear term near \( \tau = 0 \), by the coefficient in front of it is small.
Figure 3.16: a, µ, c as a function of α. Black dashed line: analytical answer obtained by solving the algebraic system (3.10.10), (3.10.15), (3.10.16), (3.10.17) numerically. Dots: numerical solution of DS equations. Red dots is βJ = 400, blue dots is βJ = 1000. In both plots J = 1.
Chapter 4

Phases and the breaking of $\mathcal{N} = 2$ supersymmetry

In chapter 2 and chapter 3, it is shown that coupled SYK/tensor models at large $N$ exhibit rich physics. In these cases, for a range of coupling, the conformal phase becomes non-unitary hence unstable. In particular, the scaling dimension of a bilinear operator becomes complex. Such an operator condenses in the infrared, and spontaneously breaks a discrete or a continuous symmetry. In both cases the symmetry that is broken is bosonic.

In this chapter we will focus on the breaking of a fermionic symmetry. To be precise, we consider a simple modification of the complex SYK model, where the Gaussian random interaction is roughly replaced by its covariance matrix. A precise version of this modification, which we will review below, was introduced by Fu, Gaiotto, Maldacena, and Sachdev in [91]. The seemingly small modification of the random Hamiltonian allows one to write it as the square of a complex (Gaussian random) operator— the combination of the conserved supercharges $Q + \tilde{Q}$, which leads to very different physics. In particular, the model in fact possesses $\mathcal{N} = 2$ supersymmetry; the $U(1)$ phase rotation of the fermions does not commute with supersymmetry and thus becomes an R-symmetry. Supersymmetry typically relates bosons and fermions via the action of $Q$, however, as noted in [91], a supersymmetric
model consisting only of fermions may be achieved as long as supersymmetry is non-linearly realized.

While the proposal in [91] seems to be the simplest generalization of SYK to incorporate \( \mathcal{N} = 2 \) supersymmetry, for reasons that will be explained shortly, we will sometimes refer to this model as \( \mathcal{N} = 2 \) SYK with fractional charges. Unlike cSYK and even \( \mathcal{N} = 1 \) versions, the model with extended supersymmetry possesses states annihilated by one complex supercharge; these BPS states have a large exact degeneracy which survives at strong coupling (this feature of the strong coupling partition function was observed in [22, 94]). It is natural to conjecture there is some relationship between a large BPS degeneracy in supersymmetric SYK and the degeneracy of supersymmetric AdS\(_2\) black holes [19, 120]. However, the \( \mathcal{N} = 2 \) SYK model with fractional charges does not have all of the expected features of these near-BPS black holes. We will come back to this point later.

The first goal of this chapter will be to perform a more exhaustive study of this \( \mathcal{N} = 2 \) SYK model, in particular at non-zero R-charge, or equivalently a non-zero chemical potential. For example, in [91] and subsequent work, the focus was placed on analyzing the model in an ensemble of zero \( U(1) \) R-charge [22, 92–98]. Because the background R-charge does not commute with supersymmetry, this choice of zero charge ensemble preserves supersymmetry and dramatically simplifies the solution of the model. We generalize this analysis and will uncover an interesting phase structure as a function of the charge when supersymmetry is broken.

We will contrast the results obtained in a non-zero R-charge sector of \( \mathcal{N} = 2 \) SYK with the solution of complex SYK. In particular we are interested in the strong coupling limit, or equivalently temperatures smaller than the scale set by the variance of the couplings. In this limit we expect an emergent conformal symmetry at the level of correlation functions between the fermions; this is the infrared reparametrization symmetry. It is spontaneously broken to \( SL(2,\mathbb{R}) \) by the associated “conformal solution” of the Schwinger-Dyson equations which determines the two-point functions. In complex SYK the scaling dimension of the fermions
is simply obtained by dimensional analysis arguments and has a universal value independent of the charge density.

In $\mathcal{N} = 2$ SYK, the model possesses not just time translation and $U(1)_R$ symmetries, but also a complex supercharge which is charged under the $U(1)_R$. In the infrared, we will show that there is an emergent super-reparametrization symmetry, that is spontaneously broken to $SU(1,1|1)$ superconformal symmetry by the conformal solution. The superconformal group $SU(1,1|1)$ extends the $SL(2,\mathbb{R})$ with bosonic subgroups $SL(2,\mathbb{R}) \times U(1)_R \subset SU(1,1|1)$. This superconformal symmetry is further broken spontaneously by the background R-charge, and because of this, the scaling dimension of the fermion is no longer fixed by dimensional analysis. Instead, one needs to look more carefully at the solution of the model to derive a constraint that uniquely determines the scaling dimension $\Delta$. The result gives a non-trivial function of the charge which can be found in equation (4.1.27) (this expression was independently derived by S. Sachdev [99]). Another feature is that the coefficient of the fermion two-point function in the conformal limit is undetermined from the IR at non-zero charge (when the charge is zero it is fully determined by supersymmetry). We will verify numerically that this parameter of the solution is fixed once the UV behavior of the model is incorporated, consistent with the idea that there is a unique two-point function at strong coupling. We also derive a Luttinger-Ward relation given in equation (4.1.50) and the grand potential in (4.1.97) (by proposing a “spooky fermion” picture for $\mathcal{N} = 2$ SYK similar to [39]), and verify numerically its validity. Finally, we analyze the spectrum of bilinear operators within the conformal solution at any charge.

This takes us to the second departure between complex and $\mathcal{N} = 2$ SYK. While the conformal solution of complex SYK is well-behaved for any charge, we find a critical charge in $\mathcal{N} = 2$ SYK for which the scaling dimension of the fermion and the overall coefficient of the two-point function vanish. At higher values of the charge, the scaling dimension becomes either negative or complex, signaling the fact that this solution is no longer physical (see for example figure 4.2). Instead, for these charges the model develops a gap and behaves as
a set of massive complex free fermions. We note that the phenomenon is distinct from the complex modes already observed in the literature such as in \([1, 2, 12]\), where some bilinear operator becomes complex. In our case, the fundamental fermion directly gets a dimension that violates unitarity.

This phase transition we found in \(\mathcal{N} = 2\) SYK is analogous to a similar phenomena observed in \([40, 100]\) for complex SYK. Those references pointed out that even in complex SYK there is a phase transition at non-zero charge where the conformal solution stops dominating and a gap develops. This leads to a new phase referred to as a “low entropy phase”. This is in contrast to the conformal phase which is “high entropy”, in the sense that there is a large order \(N\) zero temperature entropy (at least when the temperature is taken to zero after taking the large \(N\) limit, in this order). Even though there is a phase transition, it is an open problem to understand analytically the origin of this phase transition since nothing seems to go wrong with the conformal solution. Some observations on this direction were made in \([39]\) which show some unusual features in the four-point function kernel when the charge is too large, but the analysis is not conclusive and does not determine the precise value of the critical charge. In contrast we can understand this transition much better in \(\mathcal{N} = 2\) SYK. Now there is a sharp issue with the conformal solution at finite charge since the fermion itself develops an instability. We give some numerical evidence that the charge at which the fermion becomes unstable corresponds to the same charge at which there is a high- to low-entropy phase transition.

There is a nice analogy between this phase transition and black hole physics. The phase diagram described above can be compared with the one for a near extremal charged black hole, which develops an AdS$_2$ throat close to its horizon. This black hole is unstable towards discharging when a charged fermion is present with a mass (in AdS$_2$ units) smaller than the electric field, even though the black hole is perfectly stable in the vacuum. This happens through Schwinger pair production, see for example \([101]\). A similar phenomenon can also occur with scalar fields when their effective mass is below their Breitenlohner-Freedman
(BF) bound close to the horizon [102]. The supersymmetric model seems to have more in common with the candidate bulk description than complex SYK, since we find an electric field dependent scaling dimension and an instability. Thus, we can interpret the appearance of a new boundary phase at a critical charge as the endpoint of the black hole stability. Since conformal symmetry is lost, the bulk spacetime will be drastically changed and we can no longer use a dual description as a disc of Euclidean AdS$_2$ with $SL(2, \mathbb{R})$ isometry. The fact that the entropy is low in the new phase suggests that the horizon completely disappears. A schematic picture of the proposal for the bulk is in figure 4.1.

Figure 4.1: A depiction of the transition of $\mathcal{N} = 2$ SYK, along with a schematic dual bulk geometries in each phase. At background charge $Q = 0$, it was shown in [91] that the model possesses a superconformal solution. We show that for $Q < Q_{\text{crit}}$ supersymmetry may be lost (depending on which model is under consideration), but a conformal solution exists. This system is dual to a (possibly supersymmetric) AdS$_2$ geometry with a Euclidean horizon and a large extremal entropy. Schematically, a single boundary fermion two point function is computed by a charged fermion in AdS in which the semi-classical scaling dimension depends on the background charge. However, we find above a critical background charge, the dimension of the fermion violates the unitarity bound and the conformal solution ceases to be valid. In SYK, we find a new massive phase with vanishing large $N$ entropy. We conjecture the bulk dual is a horizon-less geometry.
We can now move to the second part of the chapter. So far, we have discussed the conformal solution (and when it becomes unstable). However, even though the solution of the SYK model develops a conformal symmetry in the IR, this symmetry is both spontaneously and explicitly broken. The breaking of the conformal symmetry dominates the dynamics of the model at these scales, explaining the maximal chaos for example [103]. Similarly, in $\mathcal{N} = 2$ SYK there is a supersymmetric Schwarzian mode controlling the breaking of superconformal symmetry. We verify a similar statement may be made at finite charge before the phase transition.

However, there is an essential restriction on the $\mathcal{N} = 2$ SYK model thusfar considered in the literature, and this restriction continues to the $\mathcal{N} = 2$ Schwarzian theory one derives from this model. As we mentioned previously, the model of [91] has the particular feature that the fermionic charge is fractional compared to the R-charge of the supercharge\(^a\). This leads to a Schwarzian theory with fractional charges. An interesting feature of these models of fractional charge fermions is that the BPS states come with a range of charges. This may be seen directly in the grand partition function $Z(\beta, \mu) = \text{Tr}[e^{-\beta H + \beta \mu Q}]$, and in particular can lead to cancellations in the supersymmetric index $\mathcal{I} = \text{Tr}[(-1)^F]$ which counts BPS states weighted by their fermion number. For example, in the large $N$ limit one obtains the super-Schwarzian theory which may be solved exactly—[22, 94] gives a density of states of the BPS sector for even $N$ given by

$$Z(\beta, \mu) = \sum_{Q \in \mathbb{Z}, |Q| < \frac{q}{2}} e^{\beta \mu Q} \frac{2e^{Ns_0} \cos(\frac{\pi Q}{q})}{q} + \ldots \quad (4.0.1)$$

where $s_0 = \log(2 \cos \frac{\pi}{2q})$. (For odd $N$ the same expression is valid but now the charge ranges over half-integers instead of integers.) The dots denote the non-BPS contributions to the spectrum. For even $N$ they are separated on average by a gap given by $E_{\text{gap}} = 0.412J/N$

\(^a\)In $\mathcal{N} = 2$ SYK, $q$ labels the number of fermions appearing in the supercharge, and not the number of fermions appearing in the Hamiltonian. If we normalize the supercharge to have unit R-charge, the fundamental fermion has charge $1/q$. This is consistent with charge quantization for abelian groups.
(obtained for the $q = 3$ model using the Schwarzian coupling computed in section 4.1.2). For odd $N$ the gap is exponentially small in $N$ instead. This is written with respect to the charge $Q$ that assigns unit charge to the fermions. The R-charge $Q_R$ assigning unit charge to the supercharge is related to it by $Q_R = Q/q$. The formula (4.0.1) happens to be exact for $q = 3$ and $N$ even or $N = 3 \mod 4$, but for $q > 3$ is only approximate in the large $N$ limit.

We would like to emphasize that even though we are describing the density of BPS states, their R-charge distribution (4.0.1) is not protected by supersymmetry. For example, because of the spread in charges the Witten index of this theory vanishes, something that can be reproduced by neglecting the interaction between the fermions (the index being independent of the temperature and the couplings).

As has been touched upon already, the Schwarzian theory also appears in the context of near extremal black holes in higher dimensions, describing both the classical dynamics of excitations above extremality [25,104–117] and quantum effects that become large as we take the extremal limit [18,19,118–120]. In particular, it was recently shown that a black hole in $\text{AdS}_5 \times S^5$ develops an emergent $SU(1,1|1)$ symmetry in the BPS limit in [120]. However, the Schwarzian theory describing its near-BPS dynamics is not the same as the ones constructed in [91] (Some interesting questions are being raised regarding how to understand the BPS black hole microstates in $\mathcal{N} = 2$ JT gravity in [121]). Instead, the charge of the fundamental fermion is equal to the charge of the supercharge. One can then use the same formula for the large $N$ spectrum (4.0.1) with $q = 1$, so that the BPS states have only zero R-charge and the index and degeneracy match.

Naively setting $q = 1$ does not make sense from the SYK perspective— with only fundamental fermions and a supercharge linear in them, the model would become trivial and not display any emergent conformal symmetry. To circumvent this we will instead construct models with multiple fermions\(^b\) and show one can then define interacting theories described at low energies by unit fundamental R-charge\(^c\). One interesting feature of $\mathcal{N} = 2$ SYK

\(^b\)We thank E. Witten for this suggestion.

\(^c\)In three spacetime dimensions, related models with and without supersymmetry and multiple fields
models with multiple fermions is the presence of flavor symmetries that commute with the supercharge. In this case we obtain the Schwarzian with integer R-charge and a non-zero index, in a sector of fixed flavor charge (in the simplest realization we can even gauge the flavor charge). The partition function is $Z(\beta, \mu) = e^{Ns_0} + Z_{\text{non-BPS}}(\beta, \mu)$ for some order one number $s_0$, in the Schwarzian approximation. Our new $\mathcal{N} = 2$ SYK model is thus the first of its kind to realize the $q = 1$ Schwarzian, and passing to a sector of fixed flavor charge is somewhat analogous to passing to a particular charge sector of the AdS$_5$ black holes in [120].

For completeness we study these models with multiple fermions at non-zero flavor and R-charges. We find a similar phase structure as in the models of [91] with the fundamental fermion becoming unstable at large charges. When charges are too large the conformal solution breaks down and the system is driven to a low-entropy phase, although we leave a more exhaustive analysis for future work of the phase structure of these models. We also find a potential instability as we raise the flavor charge, without breaking supersymmetry (see figure 4.13). Using the Luttinger-Ward relation we show that the conformal ansatz breaks down precisely where the charge of one of the fermions becomes maximal saturation value of $\pm N/2$.

The rest of the chapter is organized as follows. In section 4.1 we analyze the model of [91] at non-zero charge. After reviewing the definition of the model and the derivation of the mean field action we solve the IR Schwinger-Dyson equations in section 4.1.1. We discuss the breakdown of the conformal ansatz in section 4.1.2. We also solve the full Schwinger-Dyson equations numerically to verify our claims, derive a Luttinger-Ward relation for $\mathcal{N} = 2$ SYK relating the charge to the spectral asymmetry present in the IR limit of the two-point functions, find the grand potential, and compute the Schwarzian coupling numerically. In section 4.1.3 we analyze the operator spectrum by looking at the four-point function. In section 4.1.4 we elaborate on the holographic interpretation. In section 4.2 we carry out

were constructed in [122,123]. The presence of dynamical bosons in higher dimensions can lead to different conclusions about the IR effective solution compared to what is discussed in this work.
most of the same analysis for a model with multiple fermions. We analyze the behavior of the Witten index, solve the IR Schwinger-Dyson equations, study the phase structure at non-zero charge, and analyze the operator spectrum. We conclude in section 4.3 pointing out some future directions.

### 4.1 $\mathcal{N} = 2$ SYK with fractional charge

We begin with the $\mathcal{N} = 2$ supersymmetric SYK model of [91]. We will study this model at non-zero charge (some of the results were independently derived in [99]). This model consists of $N$ complex fermions which obey the standard Dirac algebra:

$$\{\psi^i, \bar{\psi}^j\} = \delta^i_j, \quad \{\psi^i, \psi^j\} = 0, \quad \{\bar{\psi}^i, \bar{\psi}^j\} = 0. \quad (4.1.1)$$

In terms of these fermions, the model is defined by a complex supercharge $Q$ with the property that it involves $q$-fermion interactions:

$$Q = i^{\frac{q-1}{2}} \sum_{1 \leq i_1 < \ldots < i_q \leq N} C_{i_1 i_2 \ldots i_q} \psi^{i_1} \psi^{i_2} \ldots \psi^{i_q}, \quad (4.1.2)$$

where we take the couplings $C_{i_1 i_2 \ldots i_q}$ to be totally antisymmetric. We introduce disorder by integrating out these couplings with Gaussian statistics and we will fix the normalization of the random variables $\langle C_{i_1 i_2 \ldots i_q} \bar{C}^{i_1 i_2 \ldots i_q} \rangle \sim J/N^{q-1}$, with a $q$-dependent constant we will fix later to match the mean field action and equations of motion (4.1.19). Additionally, we take $q$ to be odd so $Q$ has Fermi statistics. We see that (4.1.2) has $Q^2 = \bar{Q}^2 = 0$, and the Hamiltonian is

$$H = \{Q, \bar{Q}\}, \quad (4.1.3)$$

---

$^d$In contrast to [91], we employed Einstein summation rather than an explicit ordered sum $Q = i^{\frac{q-1}{2}} \sum_{1 \leq i_1 < \ldots < i_q \leq N} C_{i_1 i_2 \ldots i_q} \psi^{i_1} \psi^{i_2} \ldots \psi^{i_q}$. This leads to some different numerical factors.
as dictated by the supersymmetry algebra. When the Hamiltonian is expanded as a sum of
interactions between $2q - 2$ fermions, it has the same form as complex SYK [25, 39, 73, 124–
127], but the complex SYK couplings are not Gaussian independent variables, given instead
in terms of quadratic combinations of $C_{i_1 \ldots i_q}$.

Supersymmetry relates bosons and fermions, but with supersymmetric SYK models we
realize the supersymmetry non-linearly. For example take

$$\{ Q, \psi^i \} = 0, \quad \{ Q, \bar{\psi}_i \} = i \frac{q+1}{2} q C_{i, i_1, \ldots, i_{q-1}} \psi^{i_1} \ldots \psi^{i_{q-1}} \sim \bar{b}_i, \quad (4.1.4)$$

where the second equation defines the bosonic composite complex field $b^i$, up to a choice of
normalization. This means we do not need to include fundamental bosons in the theory in or-
der for it to be supersymmetric. It is convenient nevertheless to do a Hubbard-Stratonovich
transformation integrating-in a fundamental boson $b^i$ such that on-shell its given by the
expression above. This has two advantages. The first is that now supersymmetry transfor-
mations (4.1.4) are linearly realized. The second is that the theory with only fermions is
not melonic and is not described by a mean field action, but the equivalent theory with the
boson integrated-in is melonic and solvable at low temperatures [91].

The Hamiltonian constructed above has an additional symmetry that corresponds to
a $U(1)_R$ phase rotation of the fundamental fermions. The complex fermions transform in
conjugate representations, and therefore the supercharge is purely (anti)holomorphic with
respect to the rotation, which guarantees for example that $Q^2 = 0$. This implies that the
$U(1)_R$ symmetry is the $R$-symmetry of the one-dimensional $\mathcal{N} = 2$ Poincare algebra. The
explicit generator is:

$$Q = \sum_j \bar{\psi}_j \psi^j = \frac{N}{2}, \quad Q_R = \frac{1}{q} Q. \quad (4.1.5)$$

The first expression defines the fermion charge $Q$, with a shift of $N/2$ to enforce charge
conjugation symmetry. This is the standard definition with a fermion having unit charge.
On the right we give the natural definition of the $R$-charge $Q_R$, such that the R-charge of the
supercharge is one. In these units the fundamental fermion has a fractional R-charge $1/q$. For this reason we will refer to these modes as fractional charge $\mathcal{N} = 2$ SYK, as opposed to the models we will study in the next section.

**Refined Witten Index**

The $\mathcal{N} = 2$ SYK model is a particular instance of supersymmetric quantum mechanics, and therefore one may compute the Witten index [128] which counts the number of bosonic and fermionic ground states with a $(-1)^F = e^{-i\pi Q_R}$, with $F$ the fermion number. This is a protected quantity which may be computed in the free field limit with all $C_{i_1...i_q}$ set to zero. However, due to the presence of fractionally charged ground states with opposite bose-fermi statistics, the Witten index of this $\mathcal{N} = 2$ SYK model vanishes. As explained in [91], this model possesses a $\mathbb{Z}_q$ global symmetry under which the fermions transform with a $q$-th root of unity, and in particular commutes with the supercharge. Turning on a (discrete) chemical potential for this symmetry allows one to define a non-vanishing refined Witten index. This refined Witten index $\mathcal{I}(r) \equiv \text{Tr} \left[ (-1)^F e^{2i\pi Q_R} \right]$ for this theory is given by

$$\mathcal{I}(r) = e^{\pi i N \left( \frac{1}{2} - \frac{r}{q} \right)} (1 - e^{2\pi i r})^N = \left( 2 \sin \frac{\pi r}{q} \right)^N ,$$

which vanishes for $r = 0$. In general, the index and the number of ground states do not agree because there may be cancellation in the index. A bound on the number of ground states by computing the maximum absolute value of the index. The answer is

$$\max, \log |\mathcal{I}(r)| = N \log \left( 2 \cos \frac{\pi}{2q} \right)$$

The maximum value is given for $r_{\max} = (q \pm 1)/2$. We will match this later at large $N$ to the zero temperature entropy using the low temperature solution of the model.
4.1.1 Mean field and the conformal solution

We now review the mean field formulation of this theory. The supersymmetry can be made manifest by introducing superspace coordinates $Z \equiv (\tau, \theta, \bar{\theta})$. In terms of these coordinates, the infinitesimal supersymmetry and R-symmetry transformations are realized respectively as:

$$\tau \rightarrow \tau + \theta \bar{\eta} + \bar{\theta} \eta, \quad \theta \rightarrow \theta + \eta, \quad \bar{\theta} \rightarrow \bar{\theta} + \bar{\eta} \quad (4.1.8)$$

for an infinitesimal complex Grassmann parameter $\eta$ and,

$$\theta \rightarrow e^{ia} \theta, \quad \bar{\theta} \rightarrow e^{-ia} \bar{\theta}, \quad (4.1.9)$$

for a phase $a$. The fermion and boson (4.1.4) are encoded in a chiral fermionic superfield $\Psi^i(\tau, \theta, \bar{\theta})$, where the chirality is defined by a suitable supercovariant derivative,

$$D_{\theta} \equiv (\partial_{\theta} + \theta \partial_{\tau}), \quad D_{\theta} \Psi^i(\tau, \theta, \bar{\theta}) = 0 \implies \Psi^i(\tau, \theta, \bar{\theta}) = \psi^i(\tau + \theta \bar{\theta}) + \sqrt{2} \theta \b^i(\tau). \quad (4.1.10)$$

A similar set of expressions hold for the anti-chiral superfield $\bar{\Psi}_i(\tau, \theta, \bar{\theta})$ annihilated by $D_{\theta}$ defined by conjugation$^e$.

The supersymmetric Lagrangian density corresponding to the Hamiltonian (4.1.3) may be written in terms of the chiral superfields as

$$\mathcal{L} = \frac{1}{2} \int d^2 \theta \bar{\Psi}_i \Psi^i \psi^i + i \frac{\bar{\theta} \psi^i - \tilde{b} \bar{b}^i + i \frac{\bar{\theta} \psi^i}{2} \sqrt{2} q C_{i_1 i_2 ... i_q} \psi^{i_1} \psi^{i_2} ... \psi^{i_q} + i \frac{\psi^i}{2} \sqrt{2} q \bar{C}^{i_1 i_2 ... i_q} \bar{\psi}_{i_1} \bar{\psi}_{i_2} ... \bar{\psi}_{i_q}} \quad (4.1.11)$$

In component form, the Lagrangian becomes after integrating by parts

$$\mathcal{L} = \bar{\psi}_i \partial_{\tau} \psi^i - \bar{b}_i \b^i + i \frac{\bar{\theta} \psi^i}{2} \sqrt{2} q C_{i_1 i_2 ... i_q} \psi^{i_1} \psi^{i_2} ... \psi^{i_q} + i \frac{\psi^i}{2} \sqrt{2} q \bar{C}^{i_1 i_2 ... i_q} \bar{\psi}_{i_1} \bar{\psi}_{i_2} ... \bar{\psi}_{i_q} \quad (4.1.12)$$

$^e$In our conventions, conjugation always reverses the order of the fermions.
Following [91] we introduce the superspace two-point function $G(Z_1, Z_2) = \frac{1}{N} \langle \bar{\Psi}_i(Z_1) \Psi^i(Z_2) \rangle$.

The components of this anti-chiral-chiral superfield contains the fermion and boson two point functions (noting the sum over $i$ is implicit):

$$G_{\psi\psi}(\tau_1, \tau_2) \equiv \frac{1}{N} \langle \bar{\psi}_i(\tau_1) \psi^i(\tau_2) \rangle, \quad G_{bb}(\tau_1, \tau_2) \equiv \frac{1}{N} \langle \bar{b}_i(\tau_1) b^i(\tau_2) \rangle. \quad (4.1.13)$$

The full two-point function in superspace also includes fermionic correlators mixing fermions and bosons. The complete expansion is

$$G(Z_1, Z_2) = G_{\psi\psi}(\tau_1 - \theta_1 \bar{\theta}_1, \tau_2 + \theta_2 \bar{\theta}_2) + 2 \bar{\theta}_1 \theta_2 G_{bb}(\tau_1, \tau_2)$$

$$+ \sqrt{2} \bar{\theta}_1 G_{b\psi}(\tau_1, \tau_2 + \theta_2 \bar{\theta}_2) - \sqrt{2} \theta_2 G_{\psi b}(\tau_1 - \theta_1 \bar{\theta}_1, \tau_2). \quad (4.1.14)$$

In the large $N$ limit we will see that the classical solution of the mean field equations we derive satisfies $G_{\psi b} = 0$ and we can consistently set the mixed correlators to zero for now. It will be necessary to include them later when considering the kernel giving quadratic fluctuations around the mean field classical solution.

Starting with the action as in (4.1.11), our next goal is to produce the aforementioned mean-field equations of motion which are valid at large $N$. We will follow a series of now standard steps for SYK models, working largely in superspace:

- Integrate out the random couplings to generate a bi-local Lagrangian:

$$\mathcal{L} = \frac{1}{2} \int d^2 \theta \bar{\Psi}_i \Psi^i + \int d\theta_1 d\theta_2 \bar{\Psi}_{i_1} \bar{\Psi}_{i_2} \ldots \bar{\Psi}_{i_q} \langle C_{i_1 i_2 \ldots i_q} C_{j_1 j_2 \ldots j_q} \rangle \Psi^{j_1} \Psi^{j_2} \ldots \Psi^{j_q}. \quad (4.1.15)$$

- Integrate in the field $G(Z_1, Z_2)$ which is then fixed to be the superspace two-point function by further integrating in an anti-chiral-chiral bilocal field $\Sigma(Z_1, Z_2)$. The components of $\Sigma(Z_1, Z_2)$ contain the fermionic and bosonic self energies, $\Sigma_{\psi\psi}(\tau_1, \tau_2)$
and $\Sigma_{bb}(\tau_1, \tau_2)$, respectively. This amounts to inserting a superspace identity

$$1 = \int D\mathcal{G} D\Sigma \exp \left( -N \int d\bar{Z}_1 dZ_2 \Sigma(Z_1, Z_2) \left( \mathcal{G}(Z_1, Z_2) - \frac{1}{N} \bar{\Psi}_i(Z_1) \Psi^i(Z_2) \right) \right),$$

with the order of indices chosen to match the fact that $\mathcal{G}$ and $\Sigma$ have the same chirality.

- Integrate out the fundamental fermions to obtain an action in terms of the collective variables only. We will not write the action explicitly, but instead focus on the equations of motion.

- Vary the action with respect to the bilocal fields. To write the equations of motion in superspace, we will introduce chiral and anti-chiral integrations

$$\int dZ \equiv \int d\tau d\theta, \quad \int d\bar{Z} \equiv \int d\tau d\bar{\theta},$$

which manifest supersymmetry when acting on superfields of the appropriate chirality. Because the mean field action is bi-local, we will sometimes encounter several such integrations over distinct superspace coordinates.

The equations of motion resulting from this procedure can be written as

$$\Sigma(Z_2, Z_3) = \frac{1}{2} J \mathcal{G}(Z_2, Z_3)^{g-1},$$

$$\frac{1}{2} D_{\theta_3} \mathcal{G}(Z_1, Z_3) + \int dZ_2 \mathcal{G}(Z_1, Z_2) \Sigma(Z_3, Z_2) = \delta(Z_1 - Z_3),$$

where we define a supersymmetric delta function $\delta(Z_1 - Z_2) \equiv (\bar{\theta}_1 - \bar{\theta}_2)\delta(\tau_1 - \theta_1 \bar{\theta}_1 - \tau_2 + \theta_2 \bar{\theta}_2)$. This equation defines the normalization we chose for $J$ the coupling constant which leads to particularly simple component equations (4.1.19). Note also that the second equation is properly anti-chiral-anti-chiral in $Z_1$ and $Z_3$; in particular the superspace integration over $Z_2$ as well as the ordering of the indices inside $\mathcal{G}$ and $\Sigma$ means we do not need to introduce explicit conjugate bilocals $\bar{\mathcal{G}}$ and $\bar{\Sigma}$. 
The superspace presentation of the mean field action for the $\mathcal{N} = 2$ SYK model is well known, but as already discussed, we will also introduce and analyze the consequences of a chemical potential ($\mu$) for the $U(1)_R$ symmetry. In our conventions the chemical potential $\mu$ appears as $\mu G_{\psi\psi}$ in the action which means it couples to $qQ_R$. Because the R-symmetry does not commute with supersymmetry, or equivalently because the Grassmann variables ($\theta, \bar{\theta}$) are charged under the R-symmetry (4.1.9), the inclusion of a chemical potential generically corresponds to turning on background fields which break supersymmetry. Therefore, we will study the mean field equations in component form from now on and study the supersymmetric point as a special case.

In the fermion-number conserving saddle-point we are interested in, the mixed $G_{b\psi}$ correlators vanish and we can focus on $G_{\psi\psi}$, $G_{bb}$, and their self energies. Moreover we will also assume the solution is time translation invariant. Under these assumptions, the Schwinger-Dyson equations derived from the mean field action with chemical potential $\mu$ are given by

$$
\Sigma_{\psi\psi}(\tau) = J(q - 1)G_{\psi\psi}(\tau)^{q - 2}G_{bb}(\tau), \quad \Sigma_{bb}(\tau) = JG_{\psi\psi}(\tau)^{q - 1}, \quad (4.1.19)
$$

$$
G_{\psi\psi}(\omega) = \frac{1}{-i\omega + \mu + \Sigma_{\psi\psi}(-\omega)}, \quad G_{bb}(\omega) = \frac{1}{-1 - \Sigma_{bb}(-\omega)}. \quad (4.1.20)
$$

The $\tau = \tau_1 - \tau_2$ is the time difference and the second line is written in Fourier space. Also, one may note that while the boson $b$ carries $U(1)_R$ charge, there is no chemical potential present in the $G_{bb}$ equation because this is only an auxiliary field.

The full Schwinger-Dyson equations are complicated to solve and are typically studied numerically, as we do in section 4.1.2. However, they are tractable in the IR limit, meaning long time separations $|J\tau| \gg 1$ compared to the scale set by $J$. In this regime we will assume the theory is approximately conformal, and importantly we will see below that this assumption may be violated depending on the background charge. This conformal ansatz
determines the fermion and boson two point functions at zero temperature to be

\[ G_{\psi\psi}(\tau) = \frac{g_{\psi\psi}}{|\tau|^{2\Delta}} \left( e^{\pi \mathcal{E} \Theta(\tau)} - e^{-\pi \mathcal{E} \Theta(-\tau)} \right), \quad G_{bb}(\tau) = \frac{g_{bb}}{|\tau|^{2\Delta_b}} \left( e^{\pi \mathcal{E}_b \Theta(\tau)} + e^{-\pi \mathcal{E}_b \Theta(-\tau)} \right), \]  

(4.1.21)

where we allow for the possibility of a spectral asymmetry in order to incorporate states with non-zero charge, parametrized by \( \mathcal{E} \) for the fermion and \( \mathcal{E}_b \) for the boson. The spectral asymmetry defined in the IR should be thought of as related to the chemical potential or the charge through the UV behavior of the correlators. The prefactors \( g_{\psi\psi} \) and \( g_{bb} \) are coefficients to be determined by the equations of motion. Finally we also introduce the IR scaling dimensions for fermions \( \Delta \) and bosons \( \Delta_b \). After solving the zero temperature equations, the ansatz can also be put at finite temperature by a reparametrization

\[ G_{\psi\psi}(\tau) = g_{\psi\psi} \left( \frac{\beta}{\pi} \sin \left( \pi \frac{\tau}{\beta} \right) \right)^{-2\Delta} e^{2\pi \mathcal{E} \left( \frac{1}{2} - \frac{\tau}{\beta} \right)}, \quad G_{bb}(\tau) = g_{bb} \left( \frac{\beta}{\pi} \sin \left( \pi \frac{\tau}{\beta} \right) \right)^{-2\Delta_b} e^{2\pi \mathcal{E}_b \left( \frac{1}{2} - \frac{\tau}{\beta} \right)}, \]  

(4.1.22)

valid for \( 0 < \tau < \beta \), together with \( J \tau \gg 1 \) and \( J(\beta - \tau) \gg 1 \) so that the IR solution we are going to find is accurate.

Using the two-point functions outlined above, the next step is to solve the equations (4.1.20) in the IR strong coupling limit in which the self-energies dominate over bare propagators. This leads to the simple form \( \Sigma_{\psi\psi}(-\omega)G_{\psi\psi}(\omega) = -\Sigma_{bb}(-\omega)G_{bb}(\omega) = 1 \) (after a shift of self-energies, see [39]). The solution is obtained by transforming the two point functions to Fourier space, taking the inverse, and then transforming back to time. This determines the self-energies (which for simplicity we write at zero temperature) to be

\[ \Sigma_{\psi\psi}(\tau) = \frac{1}{g_{\psi\psi}} \frac{(1 - 2\Delta) \sin 2\pi \Delta}{2\pi \left( \cosh 2\pi \mathcal{E} + \cos 2\pi \Delta \right)} \left( -e^{\pi \mathcal{E} \Theta(-\tau)} + e^{-\pi \mathcal{E} \Theta(\tau)} \right), \]  

\[ \Sigma_{bb}(\tau) = \frac{1}{g_{bb}} \frac{(1 - 2\Delta_b) \sin 2\pi \Delta_b}{2\pi \left( \cosh 2\pi \mathcal{E}_b - \cos 2\pi \Delta_b \right)} \left( e^{\pi \mathcal{E}_b \Theta(-\tau)} + e^{-\pi \mathcal{E}_b \Theta(\tau)} \right), \]  

(4.1.23)

Having both the two point functions and self-energies, we can now plug these back into the
first set of Schwinger-Dyson equations (4.1.19). This places constraints on the dimensions, prefactors and spectral asymmetries. In contrast to the special case studied in [91] with no spectral asymmetry, we find a different set of constraints.

The equations (4.1.19) can be easily solved in steps. We can separately match the spectral asymmetry, overall scaling dimension, and prefactors in the left and right-hand sides. Matching the spectral asymmetry in both equations gives a single constraint

$$\mathcal{E}_b = -(q - 1)\mathcal{E}$$

which is consistent with the on-shell expression for the boson in term of fermions $b \sim \bar{\psi}^{q-1}$ (4.1.4). Matching scaling dimensions in both equations gives again a single constraint,

$$(q - 1)\Delta + \Delta_b = 1,$$ \hspace{1cm} (4.1.25)

which can be interpreted as demanding the interaction term $C_{i_1\ldots i_q} b^{i_1} \psi^{i_2} \ldots \psi^{i_q}$ in the component action to be marginal. For consistency of the IR ansatz, namely to make sure we can neglect the $i\omega$ and $+1$ terms in the equations (4.1.20), we need $\Delta > 0$ and $\Delta_b > 1/2$ which, upon using the constraint on dimensions (4.1.25), implies $0 < \Delta < 1/(2(q - 1))$. This means that a solution outside of this range should be considered inconsistent in the sense that the IR ansatz is invalid.

The final step to solving the equations is to match the prefactors in the left and right hand side of equations (4.1.19). The two equations give the following constraint:

$$\frac{(1 - 2\Delta) \sin 2\pi \Delta}{2\pi (\cosh 2\pi \mathcal{E} + \cos 2\pi \Delta)} = (q - 1)J g^{q-1}_{\psi\psi} g_{bb}, \quad \frac{(1 - 2\Delta_b) \sin 2\pi \Delta_b}{2\pi (\cosh 2\pi \mathcal{E}_b - \cos 2\pi \Delta_b)} = J g^{q-1}_{\bar{\psi}\bar{\psi}} g_{bb}$$ \hspace{1cm} (4.1.26)

By matching these two expressions for $g^{q-1}_{\psi\psi} g_{bb}$, we can completely determine the scaling dimensions as a function of $q$ and the spectral asymmetry $\mathcal{E}$, after using the expression for
\( \Delta_b \) and \( \mathcal{E}_b \). The solution can only be found implicitly through the equation\(^\dagger\)

\[
\frac{(1 - 2\Delta) \sin 2\pi \Delta}{\cosh 2\pi \mathcal{E} + \cos 2\pi \Delta} = (q - 1) \frac{(1 - 2(q - 1)\Delta) \sin 2\pi (q - 1)\Delta}{\cosh 2\pi (q - 1)\mathcal{E} - \cos 2\pi (q - 1)\Delta}
\]

(4.1.27)

This determines \( \Delta, \Delta_b \) and \( \mathcal{E}_b \), all as a function of \( q \) and \( \mathcal{E} \). Finally, since now the two equations (4.1.26) are identical we can only determine the combination \( g^{q-1}_{\psi\psi} g_{bb} \), but not each prefactor separately. As explained in [91] this can be traced back to an emergent symmetry in the IR Schwinger-Dyson equations under \( G_{\psi\psi}(\tau) \to \lambda G_{\psi\psi}(\tau) \) and \( G_{bb}(\tau) \to \lambda^{1-q} G_{bb}(\tau) \).

It is expected that this does not generate a relevant mode in the IR and simply the UV boundary conditions determine a precise value of \( g_{\psi\psi} \) and \( g_{bb} \). As reviewed for example in [73] unitarity requires both coefficients to be positive, and this is consistent with the constraint (4.1.26). We will verify that the full solution determines a consistent coefficient using a numerical solution of the Schwinger-Dyson equations below in section 4.1.2.

**Emergent \( SU(1,1|1) \) symmetry**

The equation above that determines \( \Delta \) cannot be solved explicitly in general. There are special values of parameters where we expect the solution to preserve supersymmetry. In these cases the equations can be exactly solved and we reproduce the results of [91] (We will see other examples in the next section where even supersymmetry is not powerful enough to fully fix the IR solution). The first observation we can make is that for the following values of the spectral asymmetry the scaling dimensions have a simple expression

\[
\mathcal{E}_{suy} = \frac{ir}{q}, \quad r \in \mathbb{Z}_q, \quad \Rightarrow \quad \Delta = \frac{1}{2q}, \quad \Delta_b = \frac{1}{2} + \frac{1}{2q}.
\]

(4.1.28)

The interpretation for this family is that it corresponds to turning on the discrete chemical potential conjugate to the \( \mathbb{Z}_q \) global symmetry that commutes with the supercharge. A simple way of deriving constrains from global supersymmetry is to construct the unique

\(^\dagger\)This expression was independently derived in [99].
superspace two-point function that is anti-chiral in the first variable, chiral in the second, and manifestly invariant under global super-translation \((\tau, \theta, \bar{\theta}) \rightarrow (\tau + \epsilon + \theta \eta + \bar{\theta} \bar{\eta}, \theta + \eta, \bar{\theta} + \bar{\eta})\), with parameters \((\epsilon, \eta, \bar{\eta})\). The answer is given by

\[ G_{\text{susy}}(Z_1, Z_2) = f(\tau_1 - \tau_2 - \theta_1 \bar{\theta}_1 - \theta_2 \bar{\theta}_2 - 2 \bar{\theta}_1 \theta_2), \quad (4.1.29) \]

where the right hand side involves an arbitrary function \(f\). Using (4.1.14) we can see that this implies \(G_{\psi\psi}(\tau_1, \tau_2) = f(\tau_1 - \tau_2)\) and \(G_{bb}(\tau_1, \tau_2) = -f'(\tau_1 - \tau_2)\). Combining these facts we obtain the constraint

\[ G_{bb}(\tau_1, \tau_2) = -\partial_{\tau_1} G_{\psi\psi}(\tau_1 - \tau_2), \quad (4.1.30) \]

which together with the conformal ansatz implies \(\Delta_b = \Delta + 1/2\), and replacing this in (4.1.19) and (4.1.20) gives \(\Delta = 1/2q\), as obtained in the equation above (4.1.28). Moreover, supersymmetry also implies a relation between coefficients \(g_{bb} = 2\Delta g_{\psi\psi}\), allowing one to fully find the solution for the prefactors as well:

\[ g_{\psi\psi}^{\text{susy}} = \left(\frac{1}{2\pi J} \frac{\sin \frac{\pi}{q}}{\cos \frac{\pi}{q} + \cos \frac{2\pi r}{q}}\right)^{1/q}, \quad g_{bb}^{\text{susy}} = \frac{1}{q} \left(\frac{1}{2\pi J} \frac{\sin \frac{\pi}{q}}{\cos \frac{\pi}{q} + \cos \frac{2\pi r}{q}}\right)^{1/q}. \quad (4.1.31) \]

These solutions correspond to a finite two-point functions except when \(r = (q \pm 1)/2\). For those two values the denominators vanish and \(g_{\psi\psi}^{\text{susy}}\) and \(g_{bb}^{\text{susy}}\) both diverge. It is interesting to note those are precisely the values at which the refined index has maximal absolute value.

So far we have imposed only the global \(\mathcal{N} = 2\) supersymmetric conditions but the model at low temperatures enjoys a bigger group of symmetries. In the IR limit we solved the Schwinger-Dyson equations (4.1.18) in a regime where we can neglect the first term and get

\[ \int dZ_2 \ G(Z_1, Z_2) \left[ \frac{J}{2} G(Z_3, Z_2)^{q-1} \right] = \delta(Z_1 - Z_3). \quad (4.1.32) \]

It was pointed out in [91] that these equations have a symmetry under \(\mathcal{N} = 2\) supersymmetric reparametrizations which we denote by \(\text{Diff}(S^{1|2})\). This consists of super-reparametrizations
\[ Z = (\tau, \theta, \bar{\theta}) \rightarrow Z' = (\tau', \theta', \bar{\theta}') \]

that satisfy the constraints

\[ D_{\bar{\theta}} \bar{\theta'} = 0 \quad D_{\theta} \tau' = \bar{\theta}' D_{\bar{\theta}} \theta', \]

\[ (4.1.33) \]

\[ D_{\bar{\theta}} \theta' = 0 \quad D_{\theta} \bar{\theta}' = \theta' D_{\bar{\theta}} \theta'. \]

\[ (4.1.34) \]

As shown in [91] the solutions of these constraints can be parametrized by a bosonic reparametrization mode \( f(\tau) \), a local \( U(1) \) transformation \( e^{ia(\tau)} \) and a complex fermionic mode \( \eta(\tau) \) which may be grouped into the bosonic and fermionic reparametrizations as:

Bosonic : \( \tau' = f(\tau), \quad \theta' = e^{ia(\tau)} \sqrt{\partial_{\tau} f(\tau)} \theta \quad \bar{\theta}' = e^{-ia(\tau)} \sqrt{\partial_{\tau} f(\tau)} \bar{\theta} \),

\[ (4.1.35) \]

Fermionic : \( \tau' = \tau + \theta \bar{\eta}(\tau) + \bar{\theta} \eta(\tau), \quad \theta' = \theta + \eta(\tau + \theta \bar{\theta}) \quad \bar{\theta}' = \bar{\theta} + \bar{\eta}(\tau - \theta \bar{\theta}) \).

\[ (4.1.36) \]

Then it’s easy to check that the IR Schwinger-Dyson equation is invariant under

\[ G(Z_1, Z_2) \rightarrow (D_{\theta_1} \theta'_1)^{\frac{1}{2}} (D_{\bar{\theta}_2} \bar{\theta}'_2)^{\frac{1}{2}} G(Z'_1, Z'_2). \]

\[ (4.1.37) \]

Even though this is a symmetry of the equations, this is not a symmetry of the solutions. The correlators found above at the supersymmetric point with \( \Delta = 1/(2q) \) are only invariant under a subgroup \( SU(1,1|1) \) of \( \text{Diff}(S^{1|2}) \). Therefore we see that the emergent super-reparametrization symmetry is spontaneously broken to the \( \mathcal{N} = 2 \) superconformal group. Of course both of these symmetries are broken by the UV term, giving rise to the \( \mathcal{N} = 2 \) Schwarzian mode.

In a state with non-zero charge, we saw that \( \Delta \) is not given by the supersymmetric solution. It is interesting to notice that the IR equations preserve supersymmetry even at finite charge, when the UV term is ignored. From this point of view, the breaking of supersymmetry that happens from the inclusion of a chemical potential is spontaneous: the solution breaks the symmetry.

From (4.1.37) one can deduce that the fermion \( \psi \) is a superconformal primary. In partic-
ular that transformation rule, applied to the solution with zero charge density only, implies the R-charge $1/q$ is twice the scaling dimension $\Delta = 1/(2q)$. This will be important to keep in mind in the next section.

It is instructive to see how a super-reparametrization explicitly acts on the two-point function. Its enough to do this at the linearized level. Bosonic reparametrizations and $U(1)_R$ transformations look the same as reparametrizations and local gauge transformations in complex SYK, and we will not repeat it here. The new generators are the infinitesimal fermion super-reparametrization, and under them the two-point function changes as

$$
\delta G_{\psi\bar{\psi}} = -G_{b\psi}(\tau_1, \tau_2)\bar{\eta}(\tau_1)
$$

(4.1.38)

$$
\delta G_{b\psi} = G_{b\psi}(\tau_1, \tau_2)\bar{\eta}(\tau_1) - \partial_{\tau_2}G_{\psi\bar{\psi}}(\tau_1, \tau_2)\bar{\eta}(\tau_2) - \frac{1}{q}G_{\psi\bar{\psi}}(\tau_1, \tau_2)\bar{\eta}'(\tau_2)
$$

(4.1.39)

$$
\delta G_{bb} = \partial_{\tau_2}G_{b\psi}(\tau_1, \tau_2)\bar{\eta}(\tau_2) - \frac{1}{q}G_{b\psi}(\tau_1, \tau_2)\bar{\eta}'(\tau_2).
$$

(4.1.40)

and a similar transformation for $\eta$. To simplify the expressions we rescale $\eta \to \eta/\sqrt{2}$ compared to (4.1.36). Its clear the first and third equation vanishes on-shell since the fermionic correlators are zero $G_{b\psi} = 0$. The second line is more interesting. On a superconformal solution of the type discussed above, the variation of $G_{b\psi}$ also vanishes whenever $\eta = \eta_0 + \eta_1 \tau$ and $\bar{\eta} = \bar{\eta}_0 + \bar{\eta}_1 \tau$. These four real parameters are the fermionic generators of $SU(1,1|1)$. For $\eta \sim \tau^n$ with $n \neq 0,1$ the transformation acts non-trivially on the two-point function, and all these modes have $SL(2,\mathbb{R})$ Casimir given by $h = 3/2$ at $\mathcal{E} = \mathcal{E}_{\text{susy}}^{\pm}$. We will see these modes later again when we compute fluctuations of the action around the classical Schwinger-Dyson solution. At the same time we see that for $\mathcal{E} \neq \mathcal{E}_{\text{susy}}$ the four fermion zero-modes are broken spontaneously since the on-shell correlators no long satisfy any supersymmetry relation.

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The Casimir operator $C_{1+2}$ acting on $\delta G_{b\psi}(\tau_1, \tau_2)$ is defined in the same way as equations (3.54) of [14], with the difference that the scaling dimension appearing in the generators of $SL(2,\mathbb{R})$ are different for the first and second insertion. Writing the eigenvalue of the Casimir as $h(h-1)$ defines the parameter $h$. 

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The Zero Temperature Entropy

We can compute the zero temperature entropy of the model and compare with the index, using the IR asymptotic of the solution of the mean field action. In order to do this it is convenient to take a derivative with respect to $q$ in the action. This simplifies when evaluating on a solution of the equations of motion and gives

$$\frac{\partial_q}{N} \log Z = J \beta \int G_{\psi\psi}(\tau) g_{bb}(\tau) \log G_{\psi\psi}(\tau),$$

(4.1.41)

$$= \# \beta + 2\pi^2 \Delta J g^{q-1}_{\psi\psi} g_{bb} + O(\beta^{-1})$$

(4.1.42)

In the second line we inserted the conformal solution, with the dots denoting terms sensitive to UV behavior. If we denote the temperature independent term in the partition function by $\log Z = \beta \# + G + O(\beta^{-1})$, where we introduce the grand potential $G$, then it is given by

$$\frac{dG}{dq} = N \frac{\pi \Delta(1 - 2\Delta) \sin 2\pi \Delta}{(q - 1)(\cos 2\pi \Delta + \cosh 2\pi \mathcal{E})}.$$  

(4.1.43)

In [73] it is explained how to go from this expression to computing $S_0(Q)$. Unfortunately this cannot be done in this model since it is not clear what boundary conditions to use when integrating over $q$. Instead, for now we will focus on the particle-hole symmetric point with $Q_R = 0$. In this case the solution is the supersymmetric one with $\mathcal{E} = 0$ which implies $\Delta = 1/(2q)$. In this case $G = S_0$ and we obtain $\frac{dS_0}{dq} = N \frac{\pi \tan \frac{\pi}{2q}}{2q^2}$. This can be easily integrated, using the free fermion limit to fix the integration constant, and gives

$$S_0(Q_R = 0) = N \log \left(2 \cos \frac{\pi}{2q}\right),$$

(4.1.44)

which precisely matches with the maximization of the index in equation (4.1.7). We will see a similar phenomenon in the models we study in the next section.

We will later propose a closed formula for the zero-temperature entropy and grand po-

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\*This expression was independently derived in [99].
potential at non-zero charge based on an extension of the spooky propagator of [39], and verify these relations numerically.

### 4.1.2 Breakdown of conformal ansatz

![Plot showing the conformal solution for the fermion scaling dimension $\Delta$ as a function of the spectral asymmetry $E$ for $q = 3$. The behavior is similar for other values of $q$.](image)

Figure 4.2: Plot showing the conformal solution for the fermion scaling dimension $\Delta$ as a function of the spectral asymmetry $E$ for $q = 3$. The behavior is similar for other values of $q$. We see the scaling dimension reaches zero at the critical spectral asymmetry, and for larger $E$, the conformal ansatz is invalid, to the right of the dashed red line.

We will now analyze what happens to the conformal solution of the Schwinger-Dyson equations as we turn on the spectral asymmetry, and therefore the background R-charge. We start with the $E = 0$ case for which we obtain a supersymmetric solution with $\Delta = \frac{1}{2q}$. As we turn on $E$ we find $\Delta$ is a solution of the transcendental equation given above. The scaling dimension smoothly goes from $\Delta(E = 0) = \frac{1}{2q}$ to $\Delta(|E| = \mathcal{E}_{\text{critical}}) = 0$. The critical asymmetry $\mathcal{E}_{\text{critical}} > 0$ is determined by the following equation

$$\frac{\sinh \pi (q - 1) \mathcal{E}_{\text{critical}}}{\cosh \pi \mathcal{E}_{\text{critical}}} = (q - 1) \tag{4.1.45}$$

For $|E| > \mathcal{E}_{\text{critical}}$ there are solution for the scaling dimension in the complex $\Delta$-plane indi-
cating the conformal ansatz breaks down. The value for \( q = 3 \) is given numerically by

\[
\mathcal{E}_{\text{critical}} = \frac{\log(1 + \sqrt{2})}{\pi} = 0.28055.. \tag{4.1.46}
\]

We show the behavior of \( \Delta \) as a function of \( \mathcal{E} \) in figure 4.2. After \( \mathcal{E} > \mathcal{E}_{\text{critical}} \), there is no valid conformal solution. In such a region, the Dyson Schwinger equations Eq.(4.1.19) can be studied numerically. We will in fact show that the solution exponentially decays once \( \mathcal{E} > \mathcal{E}_{\text{critical}} \). In contrast to known transitions [40, 100], in this case the fundamental fermion \( \psi^i \) itself develops an either negative or complex scaling, making the physical interpretation of this transition more transparent.

**Numerical Solution to Schwinger Dyson equations**

We can solve equations (4.1.19) numerically by iterations. We work in Euclidean time with finite temperature, \( \tau \sim \tau + \beta \), and for numerical purpose we consider a discretized time \( \tau_i = \frac{i\beta}{N_{\text{step}}} \). To describe continuous physics we require \( N_{\text{step}} \gg \beta J \). We work with grand canonical ensemble and fix \( \mu \). For a given \( \mu \), the expectation value of the charge can be computed by

\[
\frac{\langle Q \rangle}{N} = \frac{1}{2N} \langle [\bar{\psi}^i, \psi^i] \rangle = \frac{1}{2} (G(0) - G(\beta)) . \tag{4.1.47}
\]

We fit numerical solutions against the Ansatz given in equation (4.1.31) where we leave \( g_{\psi\psi}/g_{bb} \) and \( \mathcal{E} \) unfixed. Other parameters are fixed entirely by IR Schwinger Dyson equations. We take the best fitted \( \mathcal{E} \) as the value \( \mathcal{E}(\mu) \). Similar to the case of complex SYK, \( Q(\mu) \) and \( \mathcal{E}(\mu) \) both non-trivially depend on \( \mu \), and by tuning \( \mu \) we may understand \( Q(\mathcal{E}) \). Some examples showing the result of this procedure are presented in figure 4.3. It is evident that the conformal solution is a good approximation in the IR when \( \beta J \) is large. Moreover, the UV boundary conditions also fixes the undetermined ratio \( g_{bb}/g_{\psi\psi} \) as shown in figure 4.4.

We now turn to the determination of the charge. Similar to complex SYK it is possible

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\(^1\)The point \( \Delta = 0 \) is always a solution but one with \( g_{\psi\psi}^{q-1} g_{bb} = 0 \). Therefore at least one of the two prefactors has to vanish which is a non-physical solution.
Figure 4.3: Numerical solutions to large $N$ Dyson Schwinger equations, where from left to right $\beta J = 5, 20, 50$ and up to down $\mathcal{E} = 0, 0.15$. We plot against dimensionless quantity $\frac{2\pi \tau}{\beta}$. We also plot the best fit conformal solutions $G^c_{\psi\psi}$ and $G^c_{b\bar{b}b\bar{b}}$ and we observe good agreement up when $\tau \sim \mathcal{O}(1/J)$.

Figure 4.4: Plot of $g_{bb}/g_{\psi\psi}$ computed from the numerical solution for $q = 3$ at different values of $\mathcal{E}$. Note the infrared Schwinger-Dyson equations can only determine the combination $g_{\psi\psi}^{q-1}g_{bb}$, but the individual values are determined by the full solution. At $\mathcal{E} = 0$, it agrees with the supersymmetric answer $g_{bb} = 2\Delta g_{\psi\psi} = \frac{1}{3}g_{\psi\psi}$. 

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to derive an analytic relation between the charge and the spectral asymmetry. We do this in Appendix 1. First we define the fermionic and bosonic contribution to the total charge

\[ q_f(\Delta, \mathcal{E}) = \left( \frac{1}{2} - \Delta \right) \frac{\sinh 2\pi \mathcal{E}}{\cosh 2\pi \mathcal{E} + \cos 2\pi \Delta} + \frac{i}{2\pi} \log \left( \frac{\cos \pi (\Delta + i\mathcal{E})}{\cos \pi (\Delta - i\mathcal{E})} \right), \]  

(4.1.48)

\[ q_b(\Delta_b, \mathcal{E}_b) = \left( \frac{1}{2} - \Delta_b \right) \frac{\sinh 2\pi \mathcal{E}_b}{\cosh 2\pi \mathcal{E}_b - \cos 2\pi \Delta_b} + \frac{i}{2\pi} \log \left( \frac{\sin \pi (\Delta_b + i\mathcal{E}_b)}{\sin \pi (\Delta_b - i\mathcal{E}_b)} \right). \]  

(4.1.49)

The total fermion charge is given in terms of these functions by

\[ \frac{Q}{N} = q_f(\Delta, \mathcal{E}) + (q - 1)q_b(\Delta_b, \mathcal{E}_b). \]  

(4.1.50)

The right hand side is a function of the spectral asymmetry \( \mathcal{E} \) explicitly and through the scaling dimensions \( \Delta \) and \( \Delta_b \). This expression is the sum of two terms. The first is the contribution to the charge from the fermion itself, while the second term is a contribution to the fermion charge from the auxiliary boson. This weird behavior is due to the fact that this zeroth-order boson is an auxiliary field that has been integrated-in to simplify the interactions. This bosonic contribution is similar to the one derived in [129, 130] for first-order bosons. We have verified numerically that this relation is correct, and the result is shown in figure 4.5. In particular we can compute the charge that corresponds to the critical spectral asymmetry

\[ Q_{\text{critical}} \equiv Q(\mathcal{E}_{\text{critical}}) = 0.4142N, \]  

(4.1.51)

which is smaller than the maximal allowed value \( N/2 \). Therefore the phase transition we find is physical.

At low energy, for charges small compared to \( N \) the \( \mathcal{N} = 2 \) Schwarzian description is reliable. We can numerically determine the Super-Schwarzian coupling. Instead of directly fitting the free energy, we find it more accurate and less computationally extensive to compute the compressibility. Due to the \( \mathcal{N} = 2 \) supersymmetry, the compressibility of the \( U(1)_R \) symmetry is directly related to the super-Schwarzian coupling, as the relative
To compute the compressibility, we turn on a small chemical potential, numerically find on
and extrapolate the
We can therefore compute compressibility

\[ N \]
supersymmetry. The bosonic part of the \( N = 2 \) Super-Schwarzian is given by [22]

\[
S_b = \frac{2\pi N \alpha_S}{\beta J} \int_0^{2\pi} d\tau \left( -\text{Sch} \left( \tan \frac{\phi}{2}, \tau \right) + 2q^2 (\partial \tau a)^2 \right)
\] (4.1.52)

where \( f = \tan \phi/2 \) is the reparametrization mode on the circle and \( a(\tau) \) is the generator of the \( U(1)_R \) symmetry. We have rescaled time to have period \( 2\pi \) for convenience. The additional factor of \( q^2 \) comes from normalizing the fundamental fermion to have \( R \) charge 1 instead of \( 1/q \). On the other hand, the compressibility can be found through \( U(1)_R \) sigma model action as

\[ S_b \supset 2\pi \int_0^{2\pi} \frac{K}{2\beta} (\partial \tau a)^2. \] (4.1.53)

We can therefore compute compressibility \( K \) from the numerical Dyson-Schwinger equations and extrapolate the \( N = 2 \) Schwarzian coupling as

\[
\alpha_S = \frac{1}{4q^2} \frac{KJ}{N}. \] (4.1.54)

To compute the compressibility, we turn on a small chemical potential, numerically find on

Figure 4.5: Plot of \( Q(\mathcal{E}) \) computed with the numerical solution of the Schwinger-Dyson equations for \( q = 3 \) (blue dots). Shown in solid black line is the analytic formula (4.1.50), analogous to the Luttinger-Ward relation.
a grid of size $2^{19}$ and specialize to $q = 3$,

$$K = \left( \frac{dQ}{d\mu} \right)_{T=0} = \lim_{\mu \to 0} \frac{Q}{\mu} \approx 0.303 N/J. \quad (4.1.55)$$

This in term determines the Schwarzian coupling to be $\alpha_S \approx 0.00842$. This is different from the numerical coupling of the $\mathcal{N} = 0$ Schwarzian for complex SYK which is given by $\alpha^\text{cSYK}_S = 0.01418$ [14, 39]. As we vary $\mathcal{E}$, we can also compute the free energy and entropy. At low temperature and zero charge, the Schwarzian theory implies a free energy which admits an expansion

$$-\beta F = -\beta E_0 + S_0 + \frac{c}{2\beta}, \quad c = \frac{4\pi^2\alpha_S N}{J} \approx 0.332 \frac{N}{J}. \quad (4.1.56)$$

where $E_0$ corresponds to the ground state energy and takes a non-zero value due to normal ordering when going from the Hamiltonian to the mean field action.\footnote{We thank Yingfei Gu and Pengfei Zhang for a very useful discussion on this point.} The $S_0$ is the zero temperature entropy and the $1/\beta$ term is due to the Super-Schwarzian. In figure 4.6 we checked that the answer is consistent with a direct linear fitting against the entropy $S(\beta J)$.

We now explain how to use the numerical solution to compute the thermodynamic potentials. Since in the numerical procedure we fix $\mu$, the grand potential can be computed by the on-shell action as

$$-\beta \Omega(\mu, T)/N = \log \left( 2 \cosh \left( \frac{\beta \mu}{2} \right) \right) + \sum_\omega \log \left( \frac{1 + \frac{\Sigma_{\psi\psi}}{\omega + \mu}}{1 + \Sigma_{bb}} \right) - \beta \int_0^\beta d\tau \left. G_{\psi\psi}(\tau) \right|_{\Sigma_{\psi\psi}(\tau)},$$

\hspace{5cm} (4.1.57)

where we used the Schwinger-Dyson equations and the time translation of the solution to simplify the last term. To compute the free energy $F$, we can change to the canonical ensemble by

$$\beta F = \beta \Omega + \beta \mu Q. \quad (4.1.58)$$

As an aside, it is interesting to consider what happens with the grand potential when evaluated on supersymmetric solutions and its relation to the index, imposing periodic boundary conditions.
Figure 4.6: We checked that the answer obtained from compressibility is consistent with a direct linear fitting of the entropy $S(\beta J)$. The dots are the numerical entropy computed through numerical Dyson Schwinger solutions on a grid of size $2^{20}$. Due to finite size effects, we can not take $T$ to be too small. We show a best fit with slope fixed to be the answer determined through compressibility. The intercept which is $S_0$ is close to its analytical answer. We can also fit both the slope and the intercept, and we obtain $S(\beta J) = 0.5493 + 0.3233/(\beta J)$. The difference is about 2.7% and we conclude the answer determined both ways are consistent with each other.

conditions for the fermions and setting $\mu = 0$. The first observation we can make is that whenever a supersymmetric solution exists, which implies $\partial_\tau G_{\psi\psi} = -G_{bb}$, $\Sigma_{\psi\psi} = -\partial_\tau \Sigma_{bb}$, holds exactly, the action of the full theory is equal to the free fermion action

$$\frac{1}{N} \log Z = \log (2) + \frac{q-1}{q} \beta J (G_{\psi\psi}(\beta)^q - G_{\psi\psi}(0)^q) = \log (2). \quad (4.1.59)$$

The solutions we found numerically are at best only supersymmetric for $\tau \ll \beta$, since the thermal boundary conditions we use break supersymmetry. Even though we computed the index at zero $\mu$ we know the index vanishes, hinting that there are no classical solutions even with periodic boundary conditions. Note on the other hand, setting $\beta \mu = \frac{2\pi ir}{q}$ would automatically give the refined Witten index $\log I(r)$ in 4.1.6 at large $N$, which is not zero. It would be interesting if its possible to find the exact solution of the Schwinger-Dyson equation for $r \neq 0$ computing this quantity.

For general $\mu$ we compute the free energy and entropy numerically from (4.1.58). We
Figure 4.7: Left: Numerical solutions of $G_{\psi\psi}$ in the region where $E > E_{\text{critical}}$. We observe exponential decay solutions. Since the solution ceases to be conformal, the infrared parameter $E$ is no longer meaningful. The solutions depend on both $\beta J$ and $\mu/J$. Right: log plot of $G_{\psi\psi}$ at various values of $\beta J$, where dashed lines are linear fits. We observe that the exponent is linear in $\mu$.

show the plot of entropy in the grand canonical ensemble against chemical potential in figure 4.8. We observe a sharp transition near $\mu = J$, where the exponential solution starts to exist. The discontinuous jump in entropy is consistent with a first order transition. The transition goes from a high entropy phase to a low entropy phase, where $S_0/N$ is approximately zero.

Since SUSY is broken at non-zero $E$, the entropy $S_0$ is not protected. In fact, the system goes through a first order transition as $E$ approaches $E_{\text{critical}}$, as suggested by the results in figure 4.8. At $E > E_{\text{critical}}$, the solution exponentially decays as shown in figure 4.7. Since the solution is no longer approximately conformal, the infrared parameter $E$ ceases to be meaningful. Instead, it is more convenient to directly use the chemical potential $\mu$. At zero temperature, such exponential decay solutions can be found analytically to be

\[ G_{\psi\psi}(\tau) = e^{-(\mu-\mu_c)\tau} \Theta(\tau), \quad G_{bb}(\tau) = -\delta(\tau) + e^{((q-1)(\mu-\mu_c)+J)\tau} \Theta(-\tau), \quad (4.1.60) \]
Figure 4.8: The entropy computed numerically as a function of the chemical potential $\mu$. For each value of $\mu$, we compute the free energy and entropy on a numerical grid of size $2^{25}$ at small temperatures and extrapolate the zero temperature entropy. Note at $\mu = 0$, we obtain $S_0(0) \approx 0.5484$ which is close to the value predicted by the index $\log(2 \cos \frac{\pi}{6}) \approx 0.5493$. The transition happens slightly above $\mu = \mu_c = J$, where the exponential decaying solution (4.1.60) starts to appear. Extrapolating the precise zero temperature entropy becomes involved near the transition point.

where the critical chemical potential is determined to be

$$\mu_c = \frac{J}{2}(q - 1).$$

To verify that this is a solution of the Schwinger-Dyson equations, begin by writing the ansatz

$$G_{\psi \psi}(\tau) = e^{-(\mu - \mu_c)\tau} \Theta(\tau),$$

and use $\Sigma_{bb}(\tau) = JG_{\psi \psi}(\tau)^{q-1}$ to obtain the boson self-energy, in both position and Fourier space as

$$\Sigma_{bb}(\tau) = Je^{-(q-1)(\mu - \mu_c)\tau} \Theta(\tau), \quad \Rightarrow \quad \Sigma_{bb}(\omega) = \frac{J}{-i\omega + (q - 1)(\mu - \mu_c)}. \quad (4.1.62)$$

We can use the momentum space equation to determine $G_{bb}(\omega)$ and Fourier transform back to position space to obtain $G_{bb}(\tau) = -\delta(\tau) + e^{((q-1)(\mu - \mu_c) + J)\tau} \Theta(-\tau)$. The position space equation $\Sigma_{\psi \psi}(\tau)$ determines the value of $\mu_c$ since

$$\Sigma_{\psi \psi}(\tau) = J(q - 1)e^{-(q-2)(\mu - \mu_c)\tau} \Theta(\tau) (-\delta(\tau) + e^{((q-1)(\mu - \mu_c) + J)\tau} \Theta(-\tau)) = -\frac{J(q - 1)}{2} \delta(\tau). \quad (4.1.63)$$
where we used \( \Theta(\tau) \delta(\tau) = \frac{1}{2} \delta(\tau) \). We note that when \( J = 0 \), \( \mu_c \) is zero and the solution reduces to the retarded propagator of free massive fermion. The non-zero value \( \mu_c \) can be thought as the minimal value required for the chemical potential to overtake the random interaction and creates a gap in the spectrum. For \( q = 3 \), we note \( \mu_c = J \). Numerically a sharp transition occurs around this value.

### 4.1.3 Operator spectrum

In this section we will determine the scaling dimensions of bilinear operators at the nearly conformal fixed point. The operator spectrum provides important insights of the IR physics, in directions orthogonal to the reparametrization modes. The spectrum of the theory (4.1.3) at zero charge has been worked out and extensively discussed in [91, 92, 97]. We generalize the analysis to non-zero chemical potential, which explicitly breaks supersymmetry. However, in the infrared, supersymmetry is only spontaneously broken by the nearly conformal solution. Since the infrared equations of motions remain supersymmetric, superspace formalism provides a powerful tool to analyze the bilinear spectrum. The scaling dimensions of operators become continuous function of the chemical potential \( \mu \), or equivalently the spectral asymmetry parameter \( E \).

In particular, we will check that the spectrum does not contain an operator with a complex scaling dimension. Such a complex mode corresponds to a bulk field below its BF bound, and thus suggests an instability. Such modes are observed in various examples in nearly conformal theory [1, 2, 12]. In the presence of a complex mode, the infrared physics is not correctly described by the conformal saddle. In our case we verify that there are no complex modes in the range \( 0 \leq E < E_{\text{critical}} \), and thus the conformal solution exhibits consistent spectrum. We also discovered some unusual features in the spectrum as \( E \to E_{\text{critical}} \).

To determine the spectrum of bilinear operators, we find it convenient to look at the variation of large N Dyson Schwinger equation, and set it to zero. To see that such a
procedure determines the bilinear spectrum, we note that Dyson Schwinger equations are operator equations, and we may insert additional operators at infinity while the equations still hold. Such an insertion corresponds to a deformation

$$\delta G_{\psi\psi} = \langle \bar{\psi}_1(\tau) \psi^i(0) O_h(\infty) \rangle, \quad \delta G_{bb} = \langle \bar{b}_1(\tau) b^i(0) O_h(\infty) \rangle$$

(4.1.64)

To make such deformation non-zero, $O$ must be a bilinear operator consisting of fundamental fermions and bosons. Without losing of generality, we may restrict $O$ to be primary.

Imposing $\delta G_{\psi\psi}$ satisfies the Dyson Schwinger equation provides a necessary consistency condition for the deformation to correspond to an operator insertion, and it ‘bootstraps’ the operator spectrum. Note that this is only a necessary condition, and it is not guaranteed that all such deformation corresponds to non-trivial operators. Nontrivial operators are distinguished by their $SL(2)$ Casimir (or the supersymmetric extension) and respect the $h \to (1 - h)$ symmetry which may relate different $\delta G$.

To write the infrared Dyson Schwinger equations (4.1.32) in a more compact form, we let $G(Z_1, Z_2)^T = G(Z_2, Z_1)$ and let $(\ast, \bar{\ast})$ be chiral and anti-chiral convolutions, respectively. In this notation, the infrared equations of motion are:

$$\frac{J}{2} G^{\ast} (G^{-1})^T \equiv \int dZ_2 \, G(Z_1, Z_2) \left( \frac{1}{2} J G(Z_3; Z_2)^{q-1} \right) = (\bar{\theta}_1 - \bar{\theta}_3) \delta(\tau_1 - \tau_3 - \theta_1 \bar{\theta}_1 + \theta_3 \bar{\theta}_3).$$

(4.1.65)

$$\frac{J}{2} (G^{q-1})^{T \ast} G \equiv \int d\bar{Z}_2 \, G(Z_2, Z_1) \left( \frac{1}{2} J G(Z_2; Z_3)^{q-1} \right) = (\theta_1 - \theta_3) \delta(\tau_1 - \tau_3 + \theta_1 \bar{\theta}_1 - \theta_3 \bar{\theta}_3).$$

(4.1.66)

where $G = \langle \bar{\Psi}_i(Z_1) \Psi^i(Z_2) \rangle$ is the anti-chiral-chiral propagator, and $D_{\theta_1} G = D_{\theta_2} G = 0$. These equations are manifestly invariant under the $SU(1,1|1)$ transformation (4.1.37). However, the infrared solutions with $E \neq 0$ are not invariant under such transformations. Thus supersymmetry is only spontaneously broken in the infrared. In such a case, the fermionic transformations in $SU(1,1|1)$ transform the solution with zero $G_{\psi b}$ to solutions with non zero values of $G_{\psi b}$.  

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Vary Eq. (4.1.65) on both sides and take the convolution against $G$ from the right. Noting 
\[(A \ast B) \ast C = -A \ast (B \ast C),\] we obtain
\[\delta G(Z, Z'') = \frac{J(q - 1)}{2} \left( G \ast \left( (G^{q-2})^T \delta G^T \right) \right) \ast G,
\]
\[= \frac{J(q - 1)}{2} \int d\bar{Z}'' d\bar{Z}' G(Z, Z') G(Z'', Z')^{q-2} \delta G(Z'', Z') G(Z'', Z''') \quad (4.1.67)
\]
\[= \int d\bar{Z}'' d\bar{Z}' K^{N=2}(Z, Z''; Z'', Z') \delta G(Z'', Z') \]
In other words, $\delta G$ must be an eigenfunction of the kernel
\[K^{N=2}(Z, Z'; Z'', Z'') = \frac{J(q - 1)}{2} G(Z, Z'') G(Z'', Z') G(Z'', Z'')^{q-2}, \quad (4.1.68)
\]
with eigenvalue 1. The eigenvalue equation determines the spectrum. Note the kernel can always be written in $\mathcal{N} = 2$ superspace. If we assume full $SU(1, 1|1)$ invariance on the supercorrelator $G$, the kernel can also be diagonalized directly in the superspace [96]. In that case, the spectrum is guaranteed to organize into $SU(1, 1|1)$ supermultiplets with relative scaling dimensions fixed by symmetry. However, our conformal solution spontaneously breaks SUSY. Within a supermultiplet, the SUSY constraints on scaling dimensions no longer hold. Thus it is necessary to work in components when we diagonalize the super-kernel.

To write in components, we assume translational invariance and the supercorrelator can be expanded as:
\[G(Z_1, Z_2) = G_{\psi\bar{\psi}}(\tau_1 - \tau_2 - \theta_1 \bar{\theta}_1 - \theta_2 \bar{\theta}_2) + \sqrt{2} \bar{\theta}_1 G_{\bar{\psi}\psi}(\tau_1 - \tau_2 - \theta_2 \bar{\theta}_2)
\]
\[- \sqrt{2} \theta_2 G_{\bar{\psi}\bar{\psi}}(\tau_1 - \tau_2 - \theta_1 \bar{\theta}_1) + 2 \bar{\theta}_1 \theta_2 G_{\bar{\psi}\bar{\psi}}(\tau_1 - \tau_2), \quad (4.1.69)
\]
Now we evaluate the spectrum of the kernel Eq. (4.1.68) over the conformal saddle. In such case the vacuum carries a definite fermionic number, and
\[G(Z_1, Z_2) = G_{\psi\psi}(\tau_1 - \tau_2 - \theta_1 \bar{\theta}_1 - \theta_2 \bar{\theta}_2) + 2 \bar{\theta}_1 \theta_2 G_{\bar{\psi}\bar{\psi}}(\tau_1 - \tau_2), \quad (4.1.70)
\]
but the fluctuation does not:

$$\delta G(Z_1, Z_2) = \delta G_{\psi\psi}(\tau_1 - \tau_2 - \theta_1 \bar{\theta}_1 - \theta_2 \bar{\theta}_2) + \sqrt{2}\bar{\theta}_1 \delta G_{b\psi}(\tau_1 - \tau_2 - \theta_2 \bar{\theta}_2)$$

$$- \sqrt{2}\theta_2 \delta G_{\psi b}(\tau_1 - \tau_2 - \theta_1 \bar{\theta}_1) + 2\bar{\theta}_1 \theta_2 \delta G_{bb}(\tau_1 - \tau_2).$$  \hspace{1cm} (4.1.71)

At this point we specialize to the case of \( q = 3 \) for simplicity. Evaluating the super kernel integral in components, and matching the corresponding components in \( \delta \bar{G} \) we obtain

$$\delta G_{\psi\psi}(\tau_{12}) = -2J \int d\tau_3 d\tau_4 (G_{\psi\psi}(\tau_{14})G_{\psi\psi}(\tau_{32})G_{\psi\psi}(\tau_{34})\delta G_{bb}(\tau_{34}) + G_{\psi\psi}(\tau_{14})G_{\psi\psi}(\tau_{32})G_{bb}(\tau_{34})\delta G_{\psi\psi}(\tau_{34}))$$  \hspace{1cm} (4.1.72)

$$\delta G_{\psi b}(\tau_{12}) = 2J \int d\tau_3 d\tau_4 G_{bb}(\tau_{32})G_{\psi\psi}(\tau_{14})G_{\psi\psi}(\tau_{34})\delta G_{\psi b}(\tau_{34})$$  \hspace{1cm} (4.1.73)

$$\delta G_{bb}(\tau_{12}) = 2J \int d\tau_3 d\tau_4 G_{bb}(\tau_{14})G_{bb}(\tau_{32})G_{\psi\psi}(\tau_{34})\delta G_{\psi\psi}(\tau_{34}).$$  \hspace{1cm} (4.1.74)

To help simplify notation, we use repeated numerical indices inside the bracket to represent bilocal integral. That is

$$\text{Tr} \left( G^T \Sigma \right) = \Sigma_{AB}(12)G_{AB}(12) = \int dt_1 dt_2 \Sigma_{AB}(t_1, t_2)G_{AB}(t_1, t_2),$$  \hspace{1cm} (4.1.75)

where \( A, B \) ranges from \( \psi \) to \( b \) and trace and transpose are taken over both field indices and position space. As a first step we focus on the bosonic spectrum, where the fermionic fluctuations can be ignored. The action in the notation above becomes

$$S/N = \text{Tr} \left( \log \left( \frac{\partial_t - \Sigma_{\psi\psi}}{\partial_t - \Sigma_{bb}} \right) - G^T \Sigma \right) + \frac{J}{2} V, V = 2G_{\psi\psi}(12)^2G_{bb}(21)$$  \hspace{1cm} (4.1.76)

where the integration in \( V \) is understood. For our purpose we turned off fermionic fluctuations. We consider the quadratic fluctuations of \( S \), and note that the fluctuation over the \( J \)
\[
\delta^2 \text{Tr} \left( \log(\delta_{AB} \partial_t - \Sigma_{AB}) \right) = -\text{Tr} \left( \frac{1}{\delta \partial_t - \Sigma} \left( -\delta \Sigma \right) \frac{1}{\delta \partial_t - \Sigma} \left( -\delta \Sigma \right) \right) = -\text{Tr} \left( G^T \delta \Sigma G^T \delta \Sigma \right),
\]

(4.1.77)

where we used the equation of motion \(-\frac{1}{\delta \partial_t - \Sigma} = G^T\). The quadratic fluctuation of the term inside the trace in Eq.(4.1.76) can be thus written as

\[
\delta^2 \text{Tr} \left( \log \left( \frac{\delta_{AB} \partial_t - \Sigma_{AB}}{\delta_{AB} \partial_t - \sigma_{AB}} \right) \right) = \text{Tr} \left( G^T \delta \Sigma \delta \Sigma - G^T \delta \Sigma \delta \Sigma - 2 \delta G^T \delta \Sigma \delta \Sigma - 2 \delta G^T \delta \Sigma_{bb} \right).
\]

(4.1.78)

We can integrate over the fluctuations of the auxiliary fields \(\delta \Sigma\) and \(\delta \sigma\). The result is given by

\[
\delta G^T (k^G)^{-1} \delta G - \delta D^T (k^D)^{-1} \delta D = \\
\delta G_{AB}(12) \left( \frac{1}{G_{AD}(14)G_{CB}(32)} \right) \delta G_{CD}(34) - \delta D_{AB}(12) \left( \frac{1}{D_{AD}(14)D_{CB}(32)} \right) \delta D_{CD}(34).
\]

(4.1.79)

For \(U(1)\) neutral sector, we only turn on fluctuations along \(\delta G = (\delta G_{\psi \psi}, \delta G_{bb})\). The quadratic action takes the form

\[
\delta^2 S/N = \delta G(12) \left( \begin{array}{cc}
\frac{1}{G_{\psi \psi}(14)G_{\psi \psi}(32)} & 0 \\
0 & -\frac{1}{G_{bb}(14)G_{bb}(32)}
\end{array} \right) + 2J \left( \begin{array}{cc}
\delta_{13} \delta_{24} G_{bb}(43) & \delta_{14} \delta_{23} G_{\psi \psi}(43) \\
\delta_{14} \delta_{23} G_{\psi \psi}(34) & 0
\end{array} \right) \delta G(34)
\]

(4.1.80)
We can organize the fluctuations in the bosonic and fermionic sectors as

\[ K_b = 2J \begin{pmatrix} -G_{\psi \psi}(\tau_{14})G_{\psi \psi}(\tau_{32})G_{bb}(\tau_{34}) & -G_{\psi \psi}(\tau_{14})G_{\psi \psi}(\tau_{32})G_{\psi \psi}(\tau_{34}) \\ G_{bb}(\tau_{14})G_{bb}(\tau_{32})G_{\psi \psi}(\tau_{34}) & 0 \end{pmatrix} \] (4.1.81)

\[ K_b^* \begin{pmatrix} \delta G_{\psi \psi} \\ -\delta G_{bb} \end{pmatrix} = \begin{pmatrix} \delta G_{\psi \psi} \\ \delta G_{bb} \end{pmatrix}, \] (4.1.82)

\[ K_f = 2J G_{bb}(\tau_{32})G_{\psi \psi}(\tau_{14})G_{\psi \psi}(\tau_{34}). \] (4.1.83)

\[ \bar{K}_f = 2J G_{\psi \psi}(\tau_{32})G_{bb}(\tau_{14})G_{\psi \psi}(\tau_{34}). \] (4.1.84)

We note that the spectrum only depends on the combination \( g^2_{\psi \psi}g_{bb} \), although individual matrix element can depend on the precise value of \( g_{\psi \psi} \) and \( g_{bb} \).

To proceed we note that with non zero spectral asymmetry, the three point function of (4.1.64) can be an arbitrary linear combination of the symmetric and antisymmetric three point functions:

\[ \delta G_{\psi \psi} = \frac{A}{|\tau|^{2\Delta_1 - h}} + \frac{B \text{sgn}(\tau)}{|\tau|^{2\Delta - h}}, \quad \delta G_{bb} = \frac{a}{|\tau|^{2\Delta_b - h}} + \frac{b \text{sgn}(\tau)}{|\tau|^{2\Delta_b - h}}, \] (4.1.85)

Due to such a mixing, each matrix element in the bosonic and fermionic kernel should be thought as a 2 by 2 matrix. It is enough to work out the kernel with general spectral asymmetry, defined by

\[ K(\{\tau, \mathcal{E}, \Delta\}) = \frac{(e^{\pi\mathcal{E}_1 \Theta(\tau_{14})} - e^{-\pi\mathcal{E}_1 \Theta(\tau_{41})}) (e^{\pi\mathcal{E}_2 \Theta(\tau_{32})} - e^{-\pi\mathcal{E}_2 \Theta(\tau_{23})}) (e^{\pi\mathcal{E}_3 \Theta(\tau_{34})} - e^{-\pi\mathcal{E}_3 \Theta(\tau_{43})})}{|\tau_{14}|^{2\Delta_1} |\tau_{23}|^{2\Delta_2} |\tau_{34}|^{2\Delta_3}} \] (4.1.86)

Explicitly, by evaluating

\[ \int d\tau_3 d\tau_4 K(\{\tau, \mathcal{E}, \Delta\}) \begin{pmatrix} \frac{\text{sgn}(\tau_{14})}{|\tau_{14}|^{2\Delta_1 - h}} \\ \frac{1}{|\tau_{14}|^{2\Delta_1 - h}} \end{pmatrix} = K(\{\mathcal{E}, \Delta\}) \begin{pmatrix} \frac{\text{sgn}(\tau_{34})}{|\tau_{34}|^{2\Delta_3 - h}} \\ \frac{1}{|\tau_{34}|^{2\Delta_3 - h}} \end{pmatrix} \] (4.1.87)
Figure 4.9: The bilinear spectrum for $\mathcal{E} = 0$ and 0.15 where each intersection with horizontal axis signifies an operator with dimension corresponding to the location of the intersection. The blue curve is bosonic and the black curve is fermionic. Note $K_f$ and $\bar{K}_f$ have identical spectra, thus all fermionic curves have multiplicity 2. In addition, the presence of pairs of lines comes from the spurious doubling of the spectrum due to unphysical local symmetries. Accounting for the unphysical modes, the $\mathcal{E} = 0$ spectrum possesses the $\mathcal{N} = 2$ Schwarzian multiplet with $2 \ h = 3/2$ modes. Turning on the chemical potential leads to an IR theory with spontaneously broken supersymmetry. While the spectrum still organizes into multiplets, the $3/2$ modes are no longer protected.

we can find its matrix representation in the basis $\left( \frac{\text{sgn}(\tau)}{\sqrt{2\Delta - h}}, \frac{1}{\sqrt{2\Delta - h}} \right)$.

$$K(\{\mathcal{E}, \Delta\}) = \frac{1}{4\pi} \begin{pmatrix} c_f(\frac{3}{2} - \sum \Delta_i, 0)(\bar{Q} - Q) & c_f(\frac{3}{2} - \sum \Delta_i, 0)(P + \bar{P}) \\ c_b(\frac{3}{2} - \sum \Delta_i, 0)(Q + \bar{Q}) & -c_b(\frac{3}{2} - \sum \Delta_i, 0)(P - \bar{P}) \end{pmatrix} \quad (4.1.88)$$

where

$$c_f(\Delta, \mathcal{E}) = 2i \cos(\pi(\Delta + i\mathcal{E}))\Gamma(1 - 2\Delta), \ c_b = 2 \sin(\pi(\Delta + i\mathcal{E}))\Gamma(1 - 2\Delta), \quad (4.1.89)$$

and we introduced the parameters

$$P = c_f(\Delta_i = 1, \mathcal{E}_1)c_f(\Delta_2, \mathcal{E}_2)c_f(\Delta_3, -\mathcal{E}_3), \ P = c_f(\Delta_1, -\mathcal{E}_1)c_f(\Delta_2, -\mathcal{E}_2)c_f(\Delta_3, \mathcal{E}_3), \quad (4.1.90)$$

$$Q = c_f(\Delta_1, \mathcal{E}_1)c_f(\Delta_2, -\mathcal{E}_2)c_b(\Delta_3, \mathcal{E}_3), \ \bar{Q} = c_f(\Delta_1, -\mathcal{E}_1)c_f(\Delta_2, -\mathcal{E}_2)c_b(\Delta_3, \mathcal{E}_3). \quad (4.1.91)$$
We can look at the spectrum of the theory by evaluating (4.1.81) and (4.1.83) in terms of (4.1.88). The bosonic and fermionic spectrum are presented in Figure 4.9, where a line crossing the axis indicates the possible presence of a operator with that dimension. The fermionic operators always carry a double degeneracy as the fermionic kernels $K_f$ and $\bar{K}_f$ have identical spectra. As explained in [91], a general feature of supersymmetric SYK models is the presence of spurious modes due to new local scaling or reparametrization symmetries which act independently on each argument of the bilocals but are incompatible with the UV boundary conditions. This includes for instance a local version of the scaling discussed below (4.1.27).

At $\mathcal{E} = 0$, we observe the supermultiplet $(1, 2 \times \frac{3}{2}, 2)$ associated with the $\mathcal{N} = 2$ super Schwarzian. This contains a $h = 1$ mode corresponding to the R-symmetry, a pair of $h = 3/2$ modes indicating the presence of a complex supercharge, and a $h = 2$ mode corresponding to the Hamiltonian, or the reparametrization mode. We also see another copy of this spectrum corresponding to the spurious non-diagonal super-reparametrizations which act differently on the two arguments of the two-point function. As explained in [91], the spurious reparametrization mode is not expected to produce a soft mode in the IR, since this transformation affects the UV behavior of the correlators.

At $0 < \mathcal{E} < \mathcal{E}_{\text{critical}}$, we still observe two sets of reparametrizations modes together with the two $h = 1$ partners and two fermionic partners. Even though the field contents follow the structure of $\mathcal{N} = 2$ multiplet, the scaling dimensions do not obey the superconformal Ward identities. We find the smallest fermionic modes have $h \neq 3/2$ for $\mathcal{E} \neq 0$ due to the breaking of supersymmetry. For example at $\mathcal{E} = 0.15$, we find the four $h = \frac{3}{2}$ modes split and have scaling dimensions $h_{\pm} \approx \frac{1}{2} \pm 0.09178$. Moreover, the first supermultiplet after the Schwarzizian multiplet has dimensions $\approx (2.78478, 2 \times 3.32804, 3.77582)$ no longer obeying the superconformal constraints.

In figure 4.10 we check for the possible complex modes, whose scaling dimension takes the form $h = \frac{1}{2} + is$ determined by $SL(2, \mathbb{R})$. In all admissible range of $\mathcal{E}$ we find no complex
modes. In addition, we find no bosonic modes between $1 < h < \frac{3}{2}$, which if present could dominate over the Schwarzian in the infrared [21,131]. Therefore the bilinear spectrum is free of any known problems.

Now we turn to the behavior close to the critical asymmetry $E_{\text{critical}}$. At the critical charge, although the conformal two point coefficients (4.1.26) become zero, the kernel remains finite, and in fact equals to the identity matrix. This gives rises to unusual feature that instead of having a discrete set of modes, a continuum of modes emerges for any real value of $h$. In figure 4.11 we show the situation close but below of the transition, showing how the eigenvalue curve flattens out. Noticeably, as $E \to E_{\text{critical}}$, a continuum of complex modes also emerges as shown in figure 4.10.

### 4.1.4 Comments on the holographic interpretation

In this section we are going to make some comments towards a holographic interpretation of the features of $\mathcal{N} = 2$ SYK at finite charge, and derive a formula for the charge dependence of the zero-temperature entropy.

We begin with the most conservative point of view. Since there is a slightly broken emergent $SU(1,1|1)$ symmetry in $\mathcal{N} = 2$ SYK, and given that the dynamics is described by...
the $\mathcal{N} = 2$ Schwarzian theory, the dual two dimensional black hole is described by $\mathcal{N} = 2$ JT gravity (see for example [132]). This theory of gravity includes a $U(1)$ gauge field whose electric field generates a charge dual to the SYK charge we turned on in this chapter. The two dimensional black hole includes $N$ complex fermions $\psi_{\text{bulk}}$ which are dual to the $N$ SYK fermions $\psi^i$, and their bosonic partners. Following the notation in section 5 of [39] we take the fermions to have mass $M$ in the bulk, charge one, and they move on an electric field $E$ equal to the spectral asymmetry introduced above. When this fermion is quantized with Neumann boundary conditions the boundary two point function has the conformal form with scaling dimension $\Delta = \frac{1}{2} - \sqrt{M^2 - E^2}$. We cannot use Dirichlet boundary conditions since those would have $\Delta > 1/2$ which is not what we observe in SYK.

Assume first that the masses of the fermions do not depend on the electric field. Then the two dimensional black hole is stable as long as the electric field is not too large $E < E_{\text{critical}} = M$. Beyond the critical electric field the fermion develops a complex scaling dimension. When this happens there is a non-zero amplitude for Schwinger pair production that can screen the electric field and induce an instability of the two dimensional black hole. This is qualitatively similar to what we find in the analysis of the $\mathcal{N} = 2$ SYK fermion as a function of the spectral asymmetry with the only difference that $M$ has a dependence on $E$. The numerical analysis done for $\mathcal{N} = 2$ SYK indicates that the new phase after the instability is
not a black hole, since it does not respect the symmetries of AdS\textsubscript{2} since there is a gap.

So far the fermion coupled to JT gravity is intended to be thought of as a sector of the dual black hole to SYK. There is a different type of duality proposed in \cite{39}, which we will refer to as the ‘spooky fermion’, between some exact results in SYK and a set of fermions in the two dimensional hyperbolic disk. This proposal implies for example that the zero-temperature entropy of the complex SYK model is given by

\[ G_F(\mathcal{E}) = N[\log Z_D^F - \log Z_N^F]. \quad (4.1.92) \]

The right hand side involves the one-loop partition function of \( N \) complex fermions with Dirichlet conditions in the boundary of the hyperbolic disk, denoted by \( Z_D^F \) and \( N \) ghosts fermions with Neumann boundary conditions, denoted by \( Z_N^F \) and contributes with an extra minus sign. \( G(\Delta, \mathcal{E}) \) is the grandcanonical partition function. It was shown in \cite{39} that

\[ \frac{\partial G_F}{\partial \Delta} = -N \frac{\pi(1 - 2\Delta) \sin 2\pi \Delta}{\cosh 2\pi \mathcal{E} + \cos 2\pi \Delta}. \quad (4.1.93) \]

Then one can compute the grand canonical partition function by integrating \( \Delta \) from \( 1/2 \) to \( \Delta = 1/q \) which is the answer for complex SYK, independent of the spectral asymmetry. This identification fails for other topologies \cite{60}.

For \( \mathcal{N} = 2 \) SYK we propose a similar duality at the level of the disk topology. The first difference is the presence of a bulk boson with Dirichlet boundary conditions and a bulk ghost boson with Neumann boundary conditions. Its contribution is \( G_B(\Delta_b, \mathcal{E}_b) = N[\log Z_D^B - \log Z_N^B] \). The total grand potential is then

\[ G(\Delta, \mathcal{E}) = G_F(\Delta, \mathcal{E}) + G_B(\Delta_b, \mathcal{E}_b). \quad (4.1.94) \]

This boson is required by supersymmetry. For example at \( \mathcal{E} = 0 \) it should satisfy \( \Delta_b = \Delta + 1/2 \). Moreover, the charge of the boson has to be \((q - 1)\) times the charge of the fermion
by susy implying $\mathcal{E}_b = (q - 1)\mathcal{E}$. For $\mathcal{E} \neq 0$ susy is broken by based on the dynamics of SYK we found here we impose that $\Delta_b = 1 - (q - 1)\Delta$. It is possible to show that

\[
\frac{\partial G_B}{\partial \Delta_b} = -N \frac{\pi(1 - 2\Delta_b) \sin 2\pi\Delta_b}{\cosh 2\pi\mathcal{E}_b - \cos 2\pi\Delta_b},
\]

(4.1.95)

As opposed to complex SYK, for the $\mathcal{N} = 2$ theory the fermion dimension is not fixed. The spooky propagator gives a way of understanding the formula (4.1.27). The grand potential depends on $\Delta$ and $\mathcal{E}$. It is reasonable to expect that the value of $\Delta$ should be such that the grand potential is extremized with respect to changes in the scaling dimension. Using that $\partial_\Delta \Delta_b = -(q - 1)$ gives

\[
\partial_\Delta G(\Delta, \mathcal{E}) = 0, \Rightarrow \frac{(1 - 2\Delta) \sin 2\pi\Delta}{\cosh 2\pi\mathcal{E} + \cos 2\pi\Delta} = (q - 1) \frac{(1 - 2\Delta_b) \sin 2\pi\Delta_b}{\cosh 2\pi\mathcal{E}_b - \cos 2\pi\Delta_b},
\]

(4.1.96)

This is exactly the same equation as (4.1.27).

We can use this picture to also derive the Luttinger-Ward relation. In order to do this start from the expression for the grand potential and apply the thermodynamic relation $Q = \frac{1}{2\pi} \partial_\mathcal{E} G$ [73]. This gets contributions both from the spooky fermion and boson propagators. It is easy to check that this relation implies the Luttinger-Ward formula (4.1.50) which we verified by a numerical solution of the Schwinger-Dyson equations.

Finally, we can verify the grand potential itself matches with the one obtained from the Schwinger-Dyson equation. The explicit expression is given by

\[
G(\mathcal{E})/N = \int_\Delta^{1/2} dx \frac{\pi(1 - 2x) \sin 2\pi x}{\cosh 2\pi\mathcal{E} + \cos 2\pi x} + \int_{1/(q-1)\Delta}^{1/2} dx \frac{\pi(1 - 2x) \sin 2\pi x}{\cosh 2\pi(q - 1)\mathcal{E} - \cos 2\pi x},
\]

(4.1.97)

where $\Delta$ is the fermion scaling dimension derived from (4.1.96). Using this expression we can also compute the entropy $G = S_0(Q) - 2\pi\mathcal{E}Q$. In figure 4.12 we verify this expression coincides with the result from numerical solution of the mean field equations. We would like to stress that $G$ cannot be simply obtained from integrating (4.1.43) over $q$ since it is not
clear what boundary conditions to use for that integral, so this provides a non-trivial check.

We can verify analytically this works at $Q = 0$, giving

$$G(\mathcal{E} = 0)/N = \int_{\frac{\pi}{2\eta}}^{1/2} dx \, \pi(1 - 2x) \tan \pi x + \int_{\frac{1}{2\eta} + \frac{1}{2}}^{1/2} dx \, \pi(1 - 2x) \cot \pi x, \quad (4.1.98)$$

$$= \log \left( 2 \cos \frac{\pi}{2\eta} \right), \quad (4.1.99)$$

which coincides with the expectation from the index. This simple final answer come from non-trivial cancellations between the bosons and fermions contribution. It is possible to write down an analytic answer as a function of $\mathcal{E}$ using results of [73] but it is not very illuminating.

## 4.2 $\mathcal{N} = 2$ SYK with multiple fermions

In this section we will study models of $\mathcal{N} = 2$ SYK with multiple complex fermions. In contrast to the model introduced in [91], these new models have flavor symmetries that preserve supersymmetry. At low temperature, the IR physics is still dominated by the $\mathcal{N} = 2$
Schwarzian theory with fundamental $U(1)_R$ charge equal to the charge of the supercharge.

We will consider theories with two sets of complex fermions $\psi^i$ and $\chi^i$ with $i = 1, \ldots, N$. Due to $\mathcal{N} = 2$ supersymmetry, the Hamiltonian is completely specified by giving the supercharge. With more fermions, there is some arbitrariness in what we choose to be the supercharge which defines the model; in the simplest realization we will take it to be

$$Q = iC_{ijk} \psi^i \psi^j \chi^k,$$  \hfill (4.2.1)

where we take $C_{ijk}$ to be totally antisymmetric (this can be generalized since we only need antisymmetry in the first two indices). Then $Q^2 = \bar{Q}^2 = 0$ and $H = \{Q, \bar{Q}\}$. We take the complex coupling matrix to be a gaussian random distributed with variance $\langle C_{i_1i_2\ldots i_q} \bar{C}_{i_1i_2\ldots i_q} \rangle \sim J/N^{q-1}$. We will show these models also have an emergent $SU(1,1|1)$ symmetry at low temperatures, with additional features due to the existence of a new global symmetry.

The supersymmetric Lagrangian following from (4.2.1) has the global symmetries $U(1)_\psi \times U(1)_\chi$ which act independently on the two types of fermions $\psi$ and $\chi$. Their generators are $Q_\psi = \sum_i \bar{\psi}_i \psi^i - N/2$ and $Q_\chi = \sum_i \bar{\chi}_i \chi^i - N/2$. Both of these generators commute with the Hamiltonian but not with the supercharge. There is a linear combination that defines a supersymmetric flavor symmetry:

$$Q_F = Q_\psi - 2Q_\chi, \quad [Q, Q_F] = 0.$$ \hfill (4.2.2)

The existence of this flavor symmetry implies that deciding exactly what is the superconformal R-symmetry $U(1)_R$ inside the infrared superconformal group is a non-trivial problem. The simplest possibility is to pick $Q_\chi$ itself, but in general it could be any linear combination

$$Q_R = Q_\chi + \alpha Q_F, \quad [Q, Q_R] = Q$$ \hfill (4.2.3)
where \( \alpha \) is an arbitrary real parameter. We will determine \( \alpha \) from the solution of the model at low temperatures.

It will be useful for some calculations below to generalize the model to involve an arbitrary number of \( \psi \) fermions in the supercharge

\[
Q = i^{q-1} C_{i1...i_q} \psi^{i1} \ldots \psi^{iq-1} \chi^{iq}.
\]  

This model also has two conserved currents \( Q_\psi \) and \( Q_\chi \), and a generalization of the flavor charge (4.2.2) is

\[
Q_F = Q_\psi - (q - 1)Q_\chi.
\]  

Again, we can propose a trial low temperature R-symmetry \( Q_R = Q_\chi + \alpha Q_F \), for a parameter \( \alpha \) we will determine later.

Finally, all these models again have a global discrete \( \mathbb{Z}_q \) symmetry. This is generated by

\[
\psi \rightarrow e^{\frac{2\pi i r}{q}} \psi \quad \text{and} \quad \chi \rightarrow e^{\frac{2\pi i r}{q}} \chi
\]

and it clearly commutes with the supercharge.

**Calculation of the Index**

We will now compute the refined Witten index for this theory. The calculation can be done in the free fermion limit, but it will be useful to compare to the result we will find using the mean field description below. We should first define a notion of fermion number that produces cancellations between states when they do not preserve supersymmetry. We take the following definition:

\[
(-1)^F \equiv e^{i\pi Q_\chi}
\]  

Since this only counts the number of \( \chi \)-oscillators, it is a-priori not the ‘true’ fermion number. Nevertheless, since the supercharge involves an even number of \( \psi \) fields, all states in a supermultiplet have the same \( \psi \)-fermion number modulo 2. Furthermore, the supercharge has fermion number one with this definition and the contribution from states in the same supermultiplet will vanish. More formally, we can say that the Hilbert space breaks up into
a $\mathbb{Z}$ graded complex $\cdots \xrightarrow{\varphi} \mathcal{H}^{q_{\chi}+1} \xrightarrow{\varphi} \mathcal{H}^{q_{\chi}} \xrightarrow{\varphi} \mathcal{H}^{q_{\chi}-1} \xrightarrow{\varphi} \cdots$. This reduces mod 2 to $\mathbb{Z}_2$, the fermion number grading. There is a separate $Q$-cohomology for each $q_{\chi} \in \mathbb{Z}$ charge.

The presence of the flavor $U(1)_F$ allows us to define the Witten index refined by the chemical potential $y$ conjugate to this charge. The index is then given by

$$I(y) \equiv \text{Tr} \left[ (-1)^F e^{-\beta H} e^{iyQ_F} \right],$$

where the second line was computed in the free theory. Just like the models of [91], the index vanishes with no potential; $\text{Tr}(-1)^F = 0$.

Already at the level of the index, we may find interesting features of the model defined by (4.2.4) by passing from the grand canonical (fixed $y$) to the canonical (fixed $Q_F$) ensemble. This amounts to picking a charge sector of the theory with $Q_F \rightarrow q_F$ labeling the sector (for example $q_F = 0$ if we gauge the flavor symmetry). Now the index in this sector does not vanish and is given by

$$I(q_F) = \text{Tr}_{q_F} \left[ (-1)^F e^{-\beta H} \right],$$

where

$$I(y) \equiv \log \left( 2 \cos \left( \frac{y}{2} \right) 2 \sin \left( \frac{q-1}{2} y \right) \right).$$

Now we will take the large $N$ limit while keeping $q_F$ of order one. Therefore we can ignore the dependence on the flavor charge and focus on the $I(y)$ term to find the saddle point of the integral. In this limit the index can be approximated by

$$I \sim e^{NI(y_c)} \quad \text{where} \quad \partial_y I(y_c) = 0.$$

The saddle point equation $\partial_y I(y_c) = 0$ cannot be explicitly solved, but can be rewritten in
a form that will be useful for a later comparison

\[ \tan \left( \frac{y_c}{2} \right) = (q - 1) \cot \left( (q - 1) \frac{y_c}{2} \right), \quad \Rightarrow \quad \frac{d}{dq} \log \frac{I}{N} = \frac{y_c}{2} \cot \left( (q - 1) \frac{y_c}{2} \right). \]  

(4.2.12)

To simplify the expression for the index we took a derivative of \( I(y) \) with respect to \( q \), which due to the saddle point equation only acts on explicit \( q \) dependence. We will see below that this quantity can be identified with the zero temperature entropy \( S_0 \) computed from the mean field action at the particle-hole symmetry point.

We can also turn both a chemical potential for \( Q_F \) and for the \( \mathbb{Z}_q \) symmetry that commutes with the supercharge. The answer is given by

\[ \mathcal{I}(y, r) = \text{Tr} \left[ (-1)^F e^{-\beta H} e^{i\gamma Q_F} e^{i2\pi \frac{r}{q}(Q_F + Q)} \right], \]  

(4.2.13)

\[ = \left( 2 \cos \left( \frac{y}{2} + \frac{\pi r}{q} \right) 2 \sin \left( \frac{q - 1}{2} y - \frac{\pi r}{q} \right) \right)^N. \]  

(4.2.14)

The reason that this refinement does not add much information is that it can be rewritten as

\[ \mathcal{I}(y, r) = (-1)^{rN} \mathcal{I} \left( y - \frac{2\pi r}{q}, 0 \right), \]  

(4.2.15)

and therefore its given up to a sign by the previous index. For example the fixed charge refined index is \( \mathcal{I}(q_F, r) = e^{i2\pi r \left( \frac{N}{2} - \frac{q}{q} \right)} \mathcal{I}(q_F, 0) \). We can use this index then to determine the \( \mathbb{Z}_q \) charge of the BPS states in each fix flavor charge sector.

### 4.2.1 Mean field and the conformal solution

**Derivation of the Action**

The derivation of the mean field action is largely identical to the one in section 4.1 so we will be more brief. We will again work in superspace, first introducing introduce two chiral
superfields defined analogously to (4.1.10):

\[ \Psi^i = \psi^i (\tau + \theta \bar{\theta}) + \sqrt{2} \theta b^i_{\psi}, \quad X^i = \chi^i (\tau + \theta \bar{\theta}) + \sqrt{2} \theta b^i_{\chi}, \]

(4.2.16)

where we have again introduced auxiliary bosons \( b^i_{\psi}, b^i_{\chi} \), and their conjugates with an explicit form obtained from the analog of (4.1.4). The action after integrating in the bosons can be written in superspace in terms of these chiral fields and the interaction term is

\[ \mathcal{L} \supset i^{q-1} \int d\theta (C_{i_1i_2...i_q} \Psi^{i_1} ... \Psi^{i_{q-1}} X^{i_q}) + h.c. \]

(4.2.17)

More explicitly, the \( \theta \) component of \( C_{i_1i_2...i_q} \Psi^{i_1} ... \Psi^{i_{q-1}} X^{i_q} \) gives the combination \( \psi^{q-1} b^q_{\chi} \) and \( (q-1)b_{\psi} \psi^{q-2} \chi \). We introduce now the fermions two point function

\[ G_{\psi}(Z_1, Z_2) = \frac{1}{N} \langle \bar{\Psi}_i(Z_1) \Psi^i(Z_2) \rangle, \quad G_{\chi}(Z_1, Z_2) = \frac{1}{N} \langle \bar{X}_i(Z_1) X^i(Z_2) \rangle. \]

(4.2.18)

The expansion of these superfields includes \( G_{\psi}(\tau_1, \tau_2), G_{b_{\psi}b_{\psi}}(\tau_1, \tau_2) \), etc. To derive the mean field action we introduce a Lagrange multiplier \( \Sigma_{\psi}(Z_1, Z_2) \) and \( \Sigma_{\chi}(Z_1, Z_2) \), which in components are \( \Sigma_{\psi}(Z_1, Z_2) = \frac{1}{2} \Sigma_{b_{\psi}b_{\psi}}(\tau_1 - \theta_1 \bar{\theta}_1, \tau_2 + \theta_2 \bar{\theta}_2) + \ldots \) and similarly for \( \chi \). Next we integrate out both the couplings and the fundamental superfields. The interaction term in the mean field action involves

\[ S \supset N^J \frac{J}{2} \int dZ_1 dZ_2 \ G_{\psi}(Z_1, Z_2)^{q-1} G_{\chi}(Z_1, Z_2). \]

(4.2.19)

Since the procedure to find the action and its equations of motion (the Schwinger-Dyson equations) is similar to what we reviewed in section 4.1.1, we will move straight to the equations and solution. We consider time translation invariant solutions with vanishing
mixed correlators between bosons and fermions.

\[
\frac{1}{2} D_{\theta_3} G_\psi(Z_1, Z_3) + \int dZ_2 G_\psi(Z_1, Z_2) \left[ \frac{J}{2}(q-1) G_\psi(Z_3, Z_2) - q^2 G_\alpha(Z_3, Z_2) \right] = \delta(\bar{Z}_1 - \bar{Z}_3),
\]

(4.2.20)

\[
\frac{1}{2} D_{\theta_3} G_\chi(Z_1, Z_3) + \int dZ_2 G_\chi(Z_1, Z_2) \left[ \frac{J}{2} G_\chi(Z_3, Z_2) - q^{-1} \right] = \delta(\bar{Z}_1 - \bar{Z}_3),
\]

(4.2.21)

where the quantities in brackets are equal to \(\Sigma_\psi\) and \(\Sigma_\chi\), the superspace self-energies.

**Solutions of Schwinger-Dyson equations**

In this section, we solve the Schwinger-Dyson equations in the IR limit using the conformal ansatz, derived from the mean field action above. In position space, the component equations that determine the self energies (when off-diagonal bilinears are set to zero) are

\[
\Sigma_{\psi\psi}(\tau) = J(q - 1) \left( G_{\psi\psi}(\tau) - 2G_{b_\psi b_\psi}(\tau) \right),
\]

(4.2.22)

\[
\Sigma_{\chi\chi}(\tau) = J(q - 1)G_{b_\psi b_\psi}(\tau)G_{\psi\psi}(\tau) q^{-2},
\]

(4.2.23)

\[
\Sigma_{b_\psi b_\psi}(\tau) = JG_{\psi\psi}(\tau) q^{-1},
\]

(4.2.24)

\[
\Sigma_{b_\chi b_\chi}(\tau) = JG_{\psi\psi}(\tau) q^{-1}.
\]

(4.2.25)

We can begin by picking a similar ansatz as in the previous section for all correlators

\[
G_{AA}(\tau) = \frac{g_{AA}}{\tau^{2\Delta_A}} \left( e^{\pi E_A \Theta(\tau)} - e^{-\pi E_A \Theta(-\tau)} \right),
\]

(4.2.26)

\[
G_{b_A b_A}(\tau) = \frac{g_{b_A b_A}}{\tau^{2\Delta_{b_A}} \left( e^{\pi E_{b_A} \Theta(\tau)} + e^{-\pi E_{b_A} \Theta(-\tau)} \right),
\]

(4.2.27)

where \(A = \psi\) or \(\chi\). We now have several different scaling dimensions \(\Delta_A, \Delta_{b_A}\) and spectral asymmetries \(E_A, E_{b_A}\) for both sets of bosons and fermions.

The self-energies can be found by solving \(\Sigma_{AA}(\omega)G_{AA}(-\omega) = 1\) and \(\Sigma_{b_A b_A}(\omega)G_{b_A b_A}(-\omega) = \)
−1, and the answer is the same as before for each correlator:

\[
\Sigma_{AA}(\tau) = \frac{1}{g_{AA}} \frac{(1 - 2\Delta_A) \sin 2\pi \Delta_A}{4\pi \prod_{\pm} \cos \pi (\Delta_A \pm iE_A) |\tau|^{2(1-\Delta_A)}} \left( -e^{\pi \epsilon_A \Theta(-\tau)} + e^{-\pi \epsilon_A \Theta(\tau)} \right),
\]

\[
\Sigma_{bbA}(\tau) = \frac{1}{g_{bbA}} \frac{(1 - 2\Delta_{bbA}) \sin 2\pi \Delta_{bbA}}{4\pi \prod_{\pm} \sin \pi (\Delta_{bbA} \pm iE_{bbA}) |\tau|^{2(1-\Delta_{bbA})}} \left( e^{\pi \epsilon_{bbA} \Theta(-\tau)} + e^{-\pi \epsilon_{bbA} \Theta(\tau)} \right).
\]

We now follow the same steps as in the previous section to solve the rest of the equations. We begin by matching the left- and right-hand sides of equations (4.2.22), (4.2.23), (4.2.24), and (4.2.25). This again can be done in steps. We first match the spectral asymmetry of all equations. This gives the following two independent constraints

\[
\mathcal{E}_{b_\psi} = -(q - 2)\mathcal{E}_\psi - \mathcal{E}_\chi, \quad \mathcal{E}_{b_\chi} = -(q - 1)\mathcal{E}_\psi. \tag{4.2.28}
\]

The interpretation of these constraints is clear since integrating out auxiliary bosons gives by \( b_\psi \sim \bar{\psi}^{q-2} \bar{\chi} \) and \( b_\chi \sim \bar{\psi}^{q-1} \). Therefore their spectral asymmetry should respect these relations. Moreover, there are only two \( U(1) \) charges we are free to chose in our theory and therefore there should be only two independent spectral asymmetries which we take to be \( \mathcal{E}_\psi \) and \( \mathcal{E}_\chi \). Next, we can match the time dependence on both sides of the four equations. This gives the following two constraints on the scaling dimensions

\[
\Delta_{b_\chi} + (q - 1)\Delta_\psi = 1, \quad \Delta_{b_\psi} + (q - 2)\Delta_\psi + \Delta_\chi = 1. \tag{4.2.29}
\]

We then determine the bosonic scaling dimensions from these equations and consider \( \Delta_\psi \) and \( \Delta_\chi \) as independent variables. These constraints on the dimensions are also reasonable since they imply the interaction terms are marginal in the low energy effective action. We will see below that as opposed to Yukawa-like interactions without supersymmetry (examples of Yukawa interactions with first order bosons are \([130, 133]\)), the scaling dimensions will be determined from the spectral asymmetries uniquely.

Before moving on to matching the prefactors \( g_{AA} \) and \( g_{bbA} \) of the equations for the
self-energies, we will analyze the range of allowable scaling dimensions such that the approximations made in the IR solution are self consistent. Dropping the kinetic terms requires \( \Delta_{\text{fermion}} > 0 \) and \( \Delta_{\text{boson}} > 1/2 \), which implies the following inequalities

\[
0 < \Delta_\psi < \frac{1}{2(q-1)}, \quad 0 < \Delta_\chi < \frac{1}{2} - (q-2)\Delta_\psi < \frac{1}{2}
\]

(4.2.30)

The bound on \( \Delta_\chi \) is therefore not fixed and goes from \( 1/2 \) to \( 1/(2(q-1)) \) as we increase \( \Delta_\psi \) within the allowed range.

Now we can match the prefactors. This will produce self-consistency equations that uniquely determine the scaling dimensions. For each of the equations (4.2.22), (4.2.23), (4.2.24) and (4.2.25), we get:

\[
\frac{(1 - 2\Delta_\psi) \sin 2\pi \Delta_\psi}{4\pi \prod_{\pm} \cos \pi(\Delta_\psi \pm i\mathcal{E}_\psi)} = 2(q-1)g_{\psi\psi}^{q-1} g_{b_\psi b_\chi} + 2(q-1)(q-2) g_{b_\psi b_\psi} g_{\chi\chi} g_{\psi\psi}^{q-2}
\]

(4.2.31)

\[
\frac{(1 - 2\Delta_\chi) \sin 2\pi \Delta_\chi}{4\pi \prod_{\pm} \cos \pi(\Delta_\chi \pm i\mathcal{E}_\chi)} = 2g_{b_\chi b_\chi} g_{\psi\psi}^{q-1}
\]

(4.2.32)

\[
\frac{(1 - 2\Delta_\chi) \sin 2\pi \Delta_\chi}{4\pi \prod_{\pm} \sin \pi(\Delta_\chi \pm i\mathcal{E}_\chi)} = 2(q-1) g_{\chi\chi} g_{b_\psi b_\psi} g_{\psi\psi}^{q-2}
\]

(4.2.33)

\[
\frac{(1 - 2\Delta_\chi) \sin 2\pi \Delta_\chi}{4\pi \prod_{\pm} \sin \pi(\Delta_\chi \pm i\mathcal{E}_\chi)} = 2(q-1) g_{b_\psi b_\psi} g_{\psi\psi}^{q-2} g_{\chi\chi}
\]

(4.2.34)

These four equations are only consistent if the following two constraints are solved

\[
\frac{(1 - 2\Delta_\chi) \sin 2\pi \Delta_\chi}{4\pi \prod_{\pm} \cos \pi(\Delta_\chi \pm i\mathcal{E}_\chi)} = \frac{(1 - 2\Delta_\psi) \sin 2\pi \Delta_\psi}{4\pi \prod_{\pm} \sin \pi(\Delta_\psi \pm i\mathcal{E}_\psi)}
\]

(4.2.35)

\[
\frac{(1 - 2\Delta_\chi) \sin 2\pi \Delta_\chi}{4\pi \prod_{\pm} \cos \pi(\Delta_\chi \pm i\mathcal{E}_\chi)} = \frac{(1 - 2\Delta_\psi) \sin 2\pi \Delta_\psi}{4\pi \prod_{\pm} \sin \pi(\Delta_\psi \pm i\mathcal{E}_\psi)} + \frac{(q-2)(1 - 2\Delta_\chi) \sin 2\pi \Delta_\chi}{4\pi \prod_{\pm} \cos \pi(\Delta_\chi \pm i\mathcal{E}_\chi)}
\]

(4.2.36)

After replacing the values of scaling dimensions and spectral asymmetries for the bosons using (4.2.29) and (4.2.28), these two equations are sufficient to determine \( \Delta_\psi(\mathcal{E}_\psi, \mathcal{E}_\chi) \) and \( \Delta_\chi(\mathcal{E}_\psi, \mathcal{E}_\chi) \). We will analyze the behavior of the scaling dimensions next. First, let us point out again that using the IR solution we can only determine the following combinations of
prefactors $g_{\psi \psi}^2 g_{\chi \chi} g_{b_\psi b_\psi}$ and $g_{\psi \psi}^{-1} g_{b_\chi b_\chi}$. This is due to a set of two emergent scaling symmetries, which we refer to as $\lambda_1$ and $\lambda_2$, in the IR given by

$$
\begin{align*}
\lambda_1 : & \quad G_{\psi \psi} \to \lambda_1 G_{\psi \psi}, \quad G_{b_\psi b_\psi} \to \lambda_1^{1-q} G_{b_\psi b_\psi}, \quad G_{\chi \chi} \to \lambda_1^{2-q} G_{\chi \chi}, \quad G_{b_\psi b_\psi} \to G_{b_\psi b_\psi}, \\
\lambda_2 : & \quad G_{\psi \psi} \to G_{\psi \psi}, \quad G_{b_\psi b_\psi} \to G_{b_\psi b_\psi}, \quad G_{\chi \chi} \to \lambda_2 G_{\chi \chi}, \quad G_{b_\psi b_\psi} \to \lambda_2^{-1} G_{b_\psi b_\psi}.
\end{align*}
$$

(4.2.37)

(4.2.38)

Is it easy to see that these transformations can be obtained from acting with $U(1)_\psi$ and $U(1)_\chi$ independently on the two insertions appearing in the two-point function. This is necessary since otherwise a time independent transformation acting diagonally would leave the correlators invariant. These symmetries are broken in the UV and therefore the prefactors should be determined if we had access to the full solution. These transformations act on the UV correlators and we do not expect a low energy mode coming from them for the same reasons as in [91].

**Supersymmetric solution**

Since we find a non-vanishing index we know the ground states preserve supersymmetry. This implies for the conformal solution that supersymmetric solutions satisfy $G_{b_A b_A} \sim \partial_\tau G_{AA}$ for $A = \psi, \chi$, analogous to (4.1.30). Then the bosonic scaling dimensions are given in terms of the fermionic ones

$$
\Delta_{b_\psi} = \Delta_\psi + \frac{1}{2}, \quad \Delta_{b_\chi} = \Delta_\chi + \frac{1}{2}.
$$

(4.2.39)

As opposed to the model studied in the previous section, this does not determine the scaling dimensions anymore from a purely dimensional analysis argument. Instead we are left with a single constraint on the fermion dimensions

$$
\text{Susy : } \quad \Delta_\chi + (q - 1) \Delta_\psi = \frac{1}{2}.
$$

(4.2.40)
To determine a unique solution we need to look at the spectral asymmetry. The supersymmetry relating the bosonic and fermionic correlators also imposes a matching between the spectral asymmetries $\mathcal{E}_{b\psi} = \mathcal{E}_\psi$ and $\mathcal{E}_{b\chi} = \mathcal{E}_\chi$. Combining this with the relations (4.2.28), one finds

$$\text{Susy : } \mathcal{E}_\chi + (q - 1) \mathcal{E}_\psi = 0. \quad (4.2.41)$$

This is not the most general solution and there are discrete configurations with complex spectral asymmetry corresponding to turning on the $\mathbb{Z}_q$ charge, analogous to (4.1.28). We will not discuss those solutions here.

Using the relations above, so far supersymmetric solutions can be parametrized by $\Delta_\psi$ and $\mathcal{E}_\psi$. Given $\mathcal{E}_\psi$, we now impose the constraints coming from the prefactors in the Schwinger-Dyson equations. They give the following equation

$$\frac{\sin 2\pi \Delta_\psi}{\cos 2\pi \Delta_\psi + \cosh 2\pi \mathcal{E}_\psi} = \frac{(q - 1) \sin 2\pi (q - 1) \Delta_\psi}{\cosh 2\pi (q - 1) \mathcal{E}_\psi - \cos 2\pi (q - 1) \Delta_\psi} \quad (4.2.42)$$

This should be seen as an implicit equation determining $\Delta_\psi(\mathcal{E}_\psi)$, and all other quantities can be determined by the relations above. This is consistent with the discussion of the index, only one continuous chemical potential can be turned on while still preserving supersymmetry, the one conjugate to the UV flavor charge (4.2.2). Explicit solutions are possible for small $q$. For $q = 3$ it is given by

$$\Delta_\psi = \frac{1}{2\pi} \arcsin \left( \frac{1}{3} \sqrt{9 - \cosh^2 2\pi \mathcal{E}_\psi} \right). \quad (4.2.43)$$

We will comment later on the fact that there is a critical spectral asymmetry at which the scaling dimension vanishes.
Emergent $SU(1, 1|1)$ symmetry

We will briefly point out now that the supersymmetric solution has an emergent $SU(1, 1|1)$ symmetry. The full Schwinger-Dyson equations in superspace (4.2.20), (4.2.21) upon dropping the UV term are

$$
\int dZ G_\psi(Z_1, Z_2) (J(q-1)G_\psi(Z_3, Z_2)^q-2G_\chi(Z_3, Z_2)) = 2\delta(\bar{Z}_1 - \bar{Z}_3),
$$

(4.2.44)

$$
\int dZ G_\chi(Z_1, Z_2) (J G_\psi(Z_3, Z_2)^q-1) = 2\delta(\bar{Z}_1 - \bar{Z}_3),
$$

(4.2.45)

At late times when the UV term is ignored the equations are symmetric under the following super-reparametrizations

$$
G_\psi(Z_1, Z_2) \rightarrow (D_{\theta_1} \theta_1')^{2\Delta_\psi}(D_{\theta_2} \theta_2')^{2\Delta_\psi}G_\psi(Z'_1, Z'_2),
$$

(4.2.46)

$$
G_\chi(Z_1, Z_2) \rightarrow (D_{\theta_1} \theta_1')^{2\Delta_\chi}(D_{\theta_2} \theta_2')^{2\Delta_\chi}G_\chi(Z'_1, Z'_2),
$$

(4.2.47)

as long as

$$
(q-1)\Delta_\psi + \Delta_\chi = \frac{1}{2}.
$$

(4.2.48)

This is precisely the supersymmetric relation found above (4.2.40). This shows that the fermions are superconformal primaries under the $SU(1, 1|1)$, which is a particular case of the more general superreparametrizations above.

Since the fermions are chiral primary operators, their R-charge has to be twice their scaling dimension. This can be used to determine the parameter $\alpha$ appearing in $Q_R$ in equation (4.2.3). For example, the $\psi$ fermion has charge $Q_\psi = 1$ and $Q_\chi = 0$ and therefore $Q_R = \alpha$. The condition of $\psi$ being a chiral primary then determines the unknown coefficient $\alpha = 2\Delta_\psi$. A similar analysis for the $\chi$ fermion gives the same $\alpha$ thanks to (4.2.48).

We show later the presence of an $\mathcal{N} = 2$ Super-Schwarzian mode in the bilinear spectrum. For simplicity we can work at fixed $Q_F$ (we will consider relaxing this in the next section). Since the R-charge of the theory is given, up to a constant shift in the fixed $Q_F$ sector, by
\( Q_\chi \) this means the R-charge of the supercharge is one. The bosonic sector of the action is

\[
S_b = \frac{2\pi N \alpha_s}{\beta J} \int_0^{2\pi} d\tau \left( -\text{Sch} \left( \tan \frac{\phi}{2}, \tau \right) + 2(\partial_\tau a)^2 \right),
\]

(4.2.49)

with \( a \sim a + 2\pi \). We gave some details of the quantum mechanical spectrum of this theory in the introduction and emphasized it has the special feature that all BPS states have the same R-charge. This is consistent with the fact that the index in a fixed \( Q_F \) sector is non-vanishing and exponentially large in \( N \) (We have done some preliminary checks on these features using exact diagonalization of the model). We consider next what happens when working in an ensemble in which the flavor charge fluctuates.

**Emergent local flavor symmetry**

Just as the UV \( U(1)_R \) becomes a local IR symmetry in the conformal limit, but leads to the \( U(1)_R \) mode of the \( \mathcal{N} = 2 \) super Schwarzian, one might also expect that the \( U(1)_F \) flavor symmetry may become a spontaneously and explicitly broken local symmetry, leading to a new physical mode (and potentially a spurious mode). In the case of \( \mathcal{N} = 1 \) supersymmetry with a global \( SO(q) \) symmetry, this situation was studied in [134] where such a Kac-Moody like enhancement and corresponding flavor mode were found. In this section, we present a generalization appropriate to our model with larger supersymmetry.

While not immediately relevant for the IR Schwinger Dyson equations and the \( U(1)_F \) mode, we will briefly discuss vector multiplets and \( U(1) \) gauge transformations in superspace. A vector superfield is a real superfield with the expansion

\[
V(\tau, \theta, \bar{\theta}) = C(\tau) + \sqrt{2} \theta \lambda(\tau) - \sqrt{2} \bar{\theta} \bar{\lambda}(\tau) - \theta \bar{\theta} A(\tau), \quad V^\dagger = V.
\]

(4.2.50)

Reality ensures the \( \lambda \) are conjugates of each other and \( C, A \) are real; this condition is chosen so that \( e^{iV} \) is unitary. A supergauge transformation is represented by two bosonic chiral
superfields $\Phi(\tau, \theta, \bar{\theta})$ and $\bar{\Phi}(\tau, \theta, \bar{\theta})$:

\[
V \to V + \Phi - \bar{\Phi}, \quad \Phi(\tau, \theta, \bar{\theta}) = \phi(\tau) + \sqrt{2} \theta \xi(\tau) + \theta \bar{\theta} \partial_\tau \phi(\tau),
\]

(4.2.51)

The final term in the superfield identifies one real degree of freedom of $\phi$ with the usual gauge parameter. While we will not explore this direction further, it is interesting to note that this vector superfield would be required to introduce a supersymmetric generalization of the background chemical potential $\mu$ for the flavor symmetry.

A naive way to implement a local flavor symmetry in superspace would be to perform bi-local transformations of the form $G(Z_1, Z_2) \to e^{iV(Z_1)} e^{-iV(Z_2)} G(Z_1, Z_2)$ on the Schwinger Dyson equations (4.2.44) and (4.2.45). However, this transformation does not take chiral superfields to chiral superfields, so the correct prescription to perform bi-local independent chiral and anti-chiral gauge transformations with the correct flavor charges using (4.2.51):

\[
G_\psi(Z_1, Z_2) \to e^{-i\Phi(Z_1)} e^{i\Phi(Z_2)} G_\psi(Z_1, Z_2),
\]

(4.2.52)

\[
G_\chi(Z_1, Z_2) \to e^{i(q-1)\Phi(Z_1)} e^{-i(q-1)\Phi(Z_2)} G_\chi(Z_1, Z_2)
\]

(4.2.53)

This leaves

\[
e^{i(-\Phi(Z_1) + \Phi(Z_3))} \int dZ_2 G_\psi(Z_1, Z_2) [J(q - 1) G_\psi(Z_3, Z_2)^{q-2} G_\chi(Z_3, Z_2)] = 2\delta(\bar{Z}_1 - \bar{Z}_3),
\]

(4.2.54)

\[
e^{i(q-1)(\Phi(Z_1) - \Phi(Z_3))} \int dZ_2 G_\chi(Z_1, Z_2) [J G_\psi(Z_3, Z_2)^{q-1}] = 2\delta(\bar{Z}_1 - \bar{Z}_3),
\]

(4.2.55)

which is a symmetry of the equations of motion under the support of the supersymmetric $\delta$-function on the right hand side. Actually, the holomorphic dependence on $\Phi$ cancels inside the integral, and the resulting equations appear to have a full complex mode $\bar{\Phi}$ of transformations. This was already anticipated from (4.2.37) which was the global scaling symmetry of the conformal answer.
The above analysis indicates there are 2 bosonic and 2 fermionic flavor reparametrizations, but it is only the real part of the field $\phi(\tau)$ which leads to a compact $U(1)$ compatible with the UV picture. The imaginary part leads to a scale symmetry which we believe is not associated to any physical IR mode. It is less clear, however, that one fermionic mode should be physical and the other should be unphysical. This is because even if we take $\phi(\tau)$ to be purely real, the complex supersymmetry generators produce both $\xi$ and $\bar{\xi}$ fields, schematically:

$$
\begin{align*}
\phi &\in \mathbb{R} \\
\xi &\leftarrow \bar{\xi} \\
\partial_\tau \phi &
\end{align*}
$$

We will see in the bilinear spectrum which modes actually appear in the infrared.

Understanding now that there is one physical local $U(1)$ symmetry of the Schwinger-Dyson equations, we expect the breaking of this $U(1)$ leads to a new dynamical IR mode in addition to the $\mathcal{N} = 2$ super Schwarzian. We already reviewed the analogous IR action in Eq. (4.1.52) in which we wrote down the bosonic part of the $\mathcal{N} = 2$ Schwarzian; the $U(1)_R$ breaking boson becomes a particle moving on the $U(1)$ group manifold (with radius given in terms of the $q$ parameter). A similar effective action appears for the $U(1)_F$ mode, but there are some important differences. First, because the $U(1)_R$ is part of the super-reparametrizations, it appears in the same multiplet as the Schwarzian. In contrast, the global flavor symmetry commutes with supersymmetry, so we expect there to be a multiplet of modes in correspondence with (4.2.51). Because the group element can be written in superspace, the corresponding multiplet must describe a superparticle moving on the $U(1)_F$ group manifold.
The Zero Temperature Entropy

We now repeat the procedure in the previous section to compute the temperature independent contribution to the partition function

\[
J \beta \int G_\psi(\tau)^{q-1} G_{b'b}(\tau) \log G_\psi(\tau) + G_\psi(\tau)^{q-2} G_{b'b'}(\tau) G_{\chi\chi}(\tau)(1+(q-1) \log G_\psi(\tau)).
\]

We can evaluate the right hand side using the conformal solution. This is divergent but only the temperature independent piece is independent of the UV behavior. The answer is given by

\[
J \beta \pi \Delta_\psi \sin 2\pi(q-1) \Delta_\psi + \cosh 2\pi(q-1) \Delta_\psi - \cos 2\pi(q-1) \Delta_\psi + \cdots,
\]

where again only the temperature independence piece is insensitive to the UV behavior. Since this calculation involves only the IR behavior of the action and the solution, the expression depends only on the combination that we can determine without information about the UV.

Using equations (4.2.31) to (4.2.34), we obtain for the temperature independent piece

\[
\pi \Delta_\psi \cot \left(\pi(q-1) \Delta_\psi\right).
\]

If we restrict to supersymmetric configurations this simplifies to

\[
\frac{dS_0}{dq} = N \pi \Delta_\psi \cot (\pi(q-1) \Delta_\psi).
\]

Finally we note that if we define the zero temperature entropy as the contribution for

\[
E_\psi, E_\chi \rightarrow 0
\]

then we obtain

\[
\frac{dS_0}{dq} = N \pi \Delta_\psi \cot (\pi(q-1) \Delta_\psi).
\]
When evaluated at $E_\psi = E_\chi = 0$, the equation for $\Delta_\psi$ obtained from the Schwinger-Dyson equation becomes

$$\tan \pi \Delta_\psi = (q - 1) \cot \pi (q - 1) \Delta_\psi \quad (4.2.61)$$

As anticipated above, this is exactly the same as what we found from the index. The equation for $dS_0/dq$ and the equation for $\Delta_\psi$ matches precisely with the expression $(4.2.12)$ for the index at fixed charge, after identifying $y_c \to 2\pi \Delta_\psi$. This match can be understood as a manifestation of I-maximization in the context of quantum mechanical systems with approximate superconformal symmetry [135].

### 4.2.2 Breakdown of conformal ansatz

In this section we analyze what happens when we turn on the charge by increasing the spectral asymmetry. For large enough spectral asymmetry the conformal ansatz breaks down. We first analyze the supersymmetric solutions parametrized by a single variable, and then move on to the general case.

**Supersymmetric Solution** The physics of the supersymmetric solution is very different from the case studied in section 4.1. To begin with, there is a one-parameter family of solutions with varying scaling dimensions that are supersymmetric. Second, the coefficients in front of the Green functions are not completely determined either. Using $G_{bA} \sim \partial_\tau G_{AA}$ we can deduce that $g_{bA} = 2\Delta_A g_{AA}$ for $A = \psi, \chi$ but now the Schwinger-Dyson equations in the IR only fix the combination $g_{XX}^{q-1} g_{\psi\psi}$, while $g_{\psi\psi}$ and $g_{\chi\chi}$ cannot be independently determined without incorporating the UV behavior.

Third, we find a potential breakdown of the conformal ansatz even within the supersymmetric solution. Solving the equation above numerically we find the result shown in figure 4.13. There is again a critical spectral asymmetry $|E_\psi| = E_{\text{critical}}$ such that for $|E_\psi| > E_{\text{critical}}$ the fermion scaling dimensions are either outside the unitarity bound or complex. The
2.3 Breakdown of Conformal Ansatz

To integrate this formula we need to know $E$.

$\psi_1 = \psi_2$.

The point $2/3$ of $(2,0)$.

In this case the answer gives $E$.

$E_1 = E_2$.

What are the charges from the LW relation?

$E_q = E_3$.

The Zero Temperature Entropy

$Z$.

$E_q$.

$dS = 0$.

Then the zero-temperature 

$\Delta = 1/2q$ for a special value of $E_q$.

The critical value is implicitly given by

$$\frac{\sinh \pi (q - 1) E_{\text{critical}}}{\cosh \pi E_{\text{critical}}} = (q - 1),$$

which is exactly the same as the value found in the previous section for the model defined in [91]. As we see in the figure above, this value decreases with increasing $q$.

To determine whether this phase transition is physical or not we need to compute the charges and verify they are not maximal. We can use the Luttinger-Ward relation derived in the previous section to compute both $Q_0/N$ and $Q_1/N$. Each fermion charge includes a contribution from their respective bosonic partner, as explained in the previous section. The answer is given by

$$Q_0/N = q_f(\Delta_0, E_0) + (q - 2)q_b(\Delta_b, E_b),$$

$$Q_1/N = q_f(\Delta_1, E_1) + q_b(\Delta_b, E_b),$$

in terms of the functions defined in (4.1.48) and (4.1.49). Using these expressions for the charges we can ask now whether the critical spectral asymmetry corresponds to an instability.

Figure 4.13: Fermion scaling dimensions of the supersymmetric solutions, as a function of the spectral asymmetry of $\psi$ fermion $E_\psi$, for (a) $q = 3$ and (b) $q = 5$. We see at a finite critical value of $E_\psi$ the solution becomes singular with $\Delta_\psi \to 0$ while $\Delta_\chi \to 1/2$. This point corresponds to maximal $\chi$ charge. The two curves meet at the dashed line $\Delta = 1/2q$ for a special value of $E_\psi$. 

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or not. The first observation is that $q_f(\Delta = 1/2, \mathcal{E}) = -1/2$ and $q_b(\Delta = 1/2, \mathcal{E}) = 0$ regardless of $\mathcal{E}$. At the critical $\mathcal{E} = \mathcal{E}_{\text{critical}}$, the scaling dimension of $\chi$ and $b_\psi$ take precisely these values $\Delta_\chi(\mathcal{E}_{\text{critical}}) = 1/2$ and $\Delta_b(\mathcal{E}_{\text{critical}})$. For these reasons we obtain $|Q_\chi|(\mathcal{E}_{\text{critical}}) = N/2$. Since both fermion charges are bounded $0 \leq |Q_\psi|, |Q_\chi| \leq N/2$ this means that there is no instability in the range. The previous analysis shows that in the canonical ensemble there is no phase transition since we can turn on $Q_\chi$ and $Q_\psi$ in a supersymmetry way satisfying the constrained impose by the conformal solution. Nevertheless there can be other gapped solutions that have support in different curves in the $(Q_\psi, Q_\chi)$ plane. Then in the grand canonical ensemble there might be phase transitions between these two sets of solution. We expect this to be so, since at $\mathcal{E}_{\text{critical}}$, the flavor charge $Q_F$ is not saturated. For example, for $q = 3$, at critical spectral asymmetry, $Q_F/N = \sqrt{2} < \frac{3}{2}$. We leave a more detailed study of these possible supersymmetric phase transitions for future work.

Finally, we observe there is always a supersymmetry preserving spectral asymmetry such that $\Delta_\psi = \Delta_\chi = 1/2q$. The value of $\mathcal{E}_\psi = \mathcal{E}_\psi^*$ is determined through the following equation

$$\frac{\cos \frac{\pi}{q} + \cosh 2\pi(q - 1)\mathcal{E}_\psi^*}{\cos \frac{\pi}{q} + \cosh 2\pi\mathcal{E}_\psi^*} = (q - 1)$$

(4.2.65)

For example in the specific case $q = 3$ gives $\mathcal{E}_\psi^* = \frac{1}{2\pi} \cosh^{-1}(3/2) = 0.153...$ We can see this is consistent with the result shown in figure 4.13.

**General Behavior** In general the solution depends both on $\mathcal{E}_\psi$ and $\mathcal{E}_\chi$ independently. The behavior is similar to the one found for the supersymmetric case, when the spectral asymmetries become too large there is a breakdown of the conformal ansatz. Using the equations above one can determine the precise region in the $(\mathcal{E}_\psi, \mathcal{E}_\chi)$ plane where the emergent conformal symmetry breaks down, although we will not attempt to do it here.

**Non-conformal solution** We also propose the presence of gapped exponential decaying solutions analogous to the ones discussed in section 4.1 governing the low entropy phase. We
can see this explicitly at zero-temperature finding the solution analogous to (4.1.60). We find
\[
G_{\psi\psi}(\tau) = e^{-\mu\tau} \Theta(\tau), G_{\chi\chi}(\tau) = -e^{(2\mu+J)\tau} \Theta(-\tau),
\]
\[
G_{b\psi b\psi}(\tau) = -\delta(\tau), G_{b\chi b\chi}(\tau) = -\delta(\tau) + J e^{(2\mu+J)\tau} \Theta(-\tau).
\]
We verified that they satisfy the full Dyson Schwinger equations (4.2.22)-(4.2.25). We note that in contrast to 4.1.60, these solutions exist for any \( \mu > 0 \). We are able to verify this numerically.

4.2.3 Operator spectrum

In this section we will find the spectrum of bilinear operators of the model (4.2.1). The calculation is similar to Section 4.1.3 so we will be brief. For simplicity we specialize to the case of \( q = 3 \). The new feature of the model (4.2.1) is that we need to define a mixed super correlator between two flavors:
\[
G_{\psi X}(Z, Z') = \frac{1}{N} \langle \bar{\Psi}_i(Z) X_i(Z') \rangle = G_{\psi\chi}(t_1 - t_2 - \theta_1 \bar{\theta}_1 - \theta_2 \bar{\theta}_2) + \sqrt{2} \bar{\theta}_1 G_{b\psi\chi}(t_1 - t_2 - \theta_2 \bar{\theta}_2)
- \sqrt{2} \theta_2 G_{b\psi\chi}(t_1 - t_2 - \theta_1 \bar{\theta}_1) + 2 \bar{\theta}_1 \theta_2 G_{b\psi b\chi}(t_1 - t_2).
\]
\[
(4.2.68)
\]
Although the mixed super correlator is set to zero, their variation can be non-trivial. For convenience we will use the correlator with subscript \( G_{AB} \) to specify the supercorrelator, where \( A, B = \Psi \) or \( X \). In the conformal limit, the full equations of motions in superspace, without assuming \( G_{\psi\chi} \) to be zero, is given by
\[
G_{AB} \ast (\Sigma_{BC})^T = \delta_{AC} \delta(Z - Z''), \quad (\Sigma_{AB})^T \ast G_{BC} = \delta_{AC} \delta(Z - Z''),
\]
\[
(4.2.69)
\]
\[
\Sigma_{\psi\psi} = J \left( G_{\psi\psi} G_{XX} - G_{\psi X} G_{X\psi} \right), \quad \Sigma_{XX} = \frac{J}{2} G_{\psi\psi}^2, \quad \sigma_{\psi X} = -J G_{X\psi} G_{\psi\psi},
\]
\[
(4.2.70)
\]
where the repeated subscripts are summed over. Vary and perform convolution, and evaluating on the conformal solution \((\mathcal{G}, X)\), we obtain

\[
\delta \mathcal{G}_{\psi \psi} = J \left( \mathcal{G} \ast (\delta \mathcal{G}_{\psi \psi} X + \mathcal{G} \delta \mathcal{G}_{XX})^T \right) \bar{\ast} \mathcal{G},
\]

(4.2.71)

\[
\delta \mathcal{G}_{XX} = J \left( X \ast (\mathcal{G} \delta \mathcal{G}_{\psi \psi})^T \right) \bar{\ast} X,
\]

(4.2.72)

\[
\delta \mathcal{G}_{\psi X} = -J \left( \mathcal{G} \ast (\mathcal{G} \delta \mathcal{G}_{\psi X})^T \right) \bar{\ast} X,
\]

(4.2.73)

Along the diagonal correlators, we may define a 2 by 2 super-kernel that mixes them

\[
K^{N=2}_\text{diag} (Z_1, Z_2, Z_3, Z_4) = J \begin{pmatrix}
\mathcal{G}(Z_1, Z_4) \mathcal{G}(Z_3, Z_2) X(Z_3, Z_4) & \mathcal{G}(Z_1, Z_4) \mathcal{G}(Z_3, Z_2) \mathcal{G}(Z_3, Z_4) \\
X(Z_1, Z_4) X(Z_3, Z_2) \mathcal{G}(Z_3, Z_4) & 0
\end{pmatrix}
\]

(4.2.74)

and the off-diagonal super kernel along the directions of the mixed super correlator

\[
K^{N=2}_\text{off-diag} (Z_1, Z_2, Z_3, Z_4) = -J \mathcal{G}(Z_1, Z_4) X(Z_3, Z_2) \mathcal{G}(Z_3, Z_4).
\]

(4.2.75)

For supersymmetric solutions, super correlators only depend on the \(SU(1,1|1)\) invariant combination \(\tau_1 - \tau_2 - \theta_1 \bar{\theta}_1 - \theta_2 \bar{\theta}_2 - 2 \theta_1 \theta_2\). We present the bosonic and fermionic bilinear spectrum in figures 4.14, 4.15 and 4.16.

At \(\mathcal{E} = 0\), the leading bosonic operator has dimension \(h = 1\). In fact, we observe four \(h = 1\) modes in the bosonic spectrum. Two of them respectively correspond to the \(U(1)_R\) symmetry and \(U(1)_f\) symmetry. The other two are both spurious modes, corresponding to emergent scaling symmetry in the infrared. Together we can think in the infrared both \(U(1)_R\) and \(U(1)_f\) parameters can be taken as complex, hence accounting for the 4 degrees of freedom. We note that the two \(U(1)_f\) modes groups together with two \(h = \frac{1}{2}\) modes and becomes the BPS multiplet \((\frac{1}{2}, 1)\), whereas the two \(U(1)_R\) modes group together with the \(h = \frac{3}{2}\) and \(h = 2\) modes to form the \(\mathcal{N} = 2\) Super-Schwarzian multiplet \((1, 2 \times \frac{3}{2}, \bar{2})\) and its
Figure 4.14: The bilinear spectrum for $U(1)$ neutral sectors for $\mathcal{E} = 0$ and 0.1 where each intersection with the horizontal axis signifies an operator with dimension corresponding to the location of the intersection. The blue curve is bosonic and the black curve is fermionic. All fermionic operators are in fact doubly degenerate due to $K_f$ and $\bar{K}_f$ having identical spectrum. In addition, the presence of pairs of lines comes from the spurious doubling of the spectrum due to unphysical local symmetries. Accounting for the unphysical modes, the $\mathcal{E} = 0$ spectrum possesses the $\mathcal{N} = 2$ Schwarzian multiplet with $2h = 3/2$ modes. Turning on the chemical potential leads to an IR theory with spontaneously broken supersymmetry. While the spectrum still organizes into multiplets, the $3/2$ modes are no longer protected.

Figure 4.15: Spectrum of the $U(1)$ charged sector at distinct values of $\mathcal{E}$.

spurious partner. Hence there are in total four $h = 1$ modes and two $h = 2$ modes, and four $h = 3/2$ modes.

A new feature of the multi-fermion model is the appearance of bosonic operator in the range $1 < h < 3/2$. At $\mathcal{E} = 0$ we observe one such operator at $h \approx 1.43942$. In presence of such an operator, the $\mathcal{N} = 2$ Schwarzian becomes sub-dominant in the infrared. The leading correction becomes a bilocal action that depends on the scaling dimension of such
an operator:

\[ S \sim \int d\tau_1 d\tau_2 \left( \frac{f'(\tau_1)f' determined contribution in the low temperature expansion of free energy:

\[ -\beta F = -\beta E_0 + S_0 + \sum_{1<h<\frac{3}{2}} \frac{c_h}{\beta^{2h-2}} + \frac{c}{2\beta} + \ldots \]
Figure 4.16: Left: the bilinear spectrum for $U(1)$ neutral sectors at the enhancement point, $E_{\text{enhance}} = \frac{1}{2 \pi} \arccos \frac{3}{2}$. Right: the bilinear spectrum for $U(1)$ charged. We see that at $E = E_{\text{enhance}}$, there are additional double root at $h = 1$. Since for each charged operator $O$, $\bar{O}$ has the same dimension. There are therefore 4 additional $h = 1$ modes at the enhancement point.

4.3 Open Questions

In this chapter we have studied the model of [91] at non-zero charge and proposed new models of $\mathcal{N} = 2$ SYK with interesting features. We conclude with some open questions and future directions.

A future direction is to develop, for the models of [91], a detailed picture for the bulk gravity dual description. In particular we would like to find a good model for the bulk interpretation of the low-entropy phase, with the hope of discovering new features of the instabilities of near extremal black holes in higher dimensions. Moreover, we have given a simple description of this transition in $\mathcal{N} = 2$ SYK, since the fundamental SYK fermion becomes unstable. As far as we know, it is an open problem to extend this kind of picture to complex SYK, where the origin of this transition is more mysterious from the perspective of the conformal phase (some comments in this direction are made in [39] looking at the bilinear spectrum as a function of the charge).

For the SYK model of [91] we obtained the $\mathcal{N} = 2$ Schwarzian coupling at background
zero charge. It would be interesting to compute the Schwarzian coupling as a function
of the charge. In this case it would not have a relation with the $U(1)_R$ compressibility.
Instead the low energy mode would be closer to the one of complex SYK since the fermionic
modes of the Super-Schwarzian would become massive. An exhaustive analysis of the exact
diagonalization of these models would help to understand the nearly conformal sector of the
spectrum, as well as the transition when this picture breaks down. Relatedly, we have not
explored exhaustively the phase diagram of the $\mathcal{N} = 2$ SYK model with multiple fermions
as a function of an arbitrary pair of $U(1)$ charges, and even a numerical solution of the mean
field equations are complicated to obtain.

Finally we constructed models that develop $\mathcal{N} = 2$ Schwarzian theories with funda-
mental R-charge one. This presents features in the spectrum that are the closest to the
$\mathcal{N} = 4$ Schwarzian theory describing BPS black holes in flat space [19], as explained in the
introduction. We hope the model introduced in section 4.2 can help find such theories with
$\mathcal{N} = 4$ supersymmetry when considering multiple fermions transforming in representations
of $SU(2)$ without the need of including second order bosons [136]. We leave this for future
work.

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Here we derive the relation between the charge density of our models and the spectral asymmetry of the IR correlators. We follow the derivation presented in section 2 of [39], based on similar ideas considered in Appendix C of [137].

We begin by deriving the Luttinger-Ward relation to the model of section 4.1, introduced in [91]. The first step is to define a notion of “flow” of the green functions $G$ and $D$, introducing a bilocal conserved current $j(\tau_1, \tau_2)$. Begin by generalizing the UV contributions of the mean field action from $\delta'(\tau_1 - \tau_2) - \mu \delta(\tau_1 - \tau_2) \to \mu_f(\tau_1, \tau_2)$ for fermions and $\delta(\tau_1 - \tau_2) \to \mu_b(\tau_1 - \tau_2)$ for bosons, and defining the new self energies $\tilde{\Sigma}_{\psi\psi}(\tau_1, \tau_2) = \Sigma_{\psi\psi}(\tau_1, \tau_2) + \mu_f(\tau_1, \tau_2)$ and $\tilde{\Sigma}_{bb}(\tau_1, \tau_2) = \Sigma_{bb}(\tau_1, \tau_2) + \mu_b(\tau_1, \tau_2)$. Under this transformation we end up with a mean field action

$$I[G, \Sigma] = I_{IR}[G, \tilde{\Sigma}] + \int d\tau_1 d\tau_2 \left[ \mu_f(\tau_1, \tau_2)G_{\psi\psi}(\tau_2, \tau_1) + \mu_b(\tau_1, \tau_2)G_{bb}(\tau_2, \tau_1) \right], \quad (1.1)$$

where $I_{IR}$ is an action such that its equation of motion are exactly the Schwinger-Dyson ones in the IR approximation. Now define the following bilocal current

$$j(\tau_1, \tau_2) = \mu(\tau_1, \tau_2)G_{\psi\psi}(\tau_2, \tau_1) + (q - 1)\mu_b(\tau_1, \tau_2)G_{bb}(\tau_2, \tau_1) - (1 \leftrightarrow 2), \quad (1.2)$$

$$= \tilde{\Sigma}_{\psi\psi}(\tau_1, \tau_2)G_{\psi\psi}(\tau_2, \tau_1) + (q - 1)\tilde{\Sigma}_{bb}(\tau_1, \tau_2)G_{bb}(\tau_2, \tau_1) - (1 \leftrightarrow 2). \quad (1.3)$$

Now we can exploit the symmetries of the IR equations to say something about this bilocal current. The IR equations coming from $I_{IR}$ are invariant under the local $U(1)$ transformations

$$G_{\psi\psi}(\tau_1, \tau_2) \to e^{i\lambda(\tau_1)}e^{-i\lambda(\tau_2)}G_{\psi\psi}(\tau_1, \tau_2), \quad G_{bb}(\tau_1, \tau_2) \to e^{i(q-1)\lambda(\tau_1)}e^{-i(q-1)\lambda(\tau_2)}G_{bb}(\tau_1, \tau_2). \quad (1.4)$$

Imitating the derivation in [39] we can use this symmetry to show that, evaluated on a
classical solution of the mean field action, the bilocal current satisfies the local conservation through the vanishing of \( \int_{-\infty}^{+\infty} j(\tau_1, \tau_0) d\tau_1 = 0 \). Then we can define the charge as

\[
\tilde{Q} = \int_{-\infty}^{\tau_0} d\tau_1 \int_{\tau_0}^{\infty} d\tau_2 \, j(\tau_1, \tau_2). \tag{1.5}
\]

We can verify using the UV behavior of the Green functions that this coincides with the charge of the \( \psi \) fermion when we choose \( \mu_f \to \delta'(\tau_1 - \tau_2) - \mu \delta(\tau_1 - \tau_2) \) and \( \mu_b \to \delta(\tau_1 - \tau_2) \), then we get

\[
\tilde{Q} = - \int_{-\infty}^{\infty} d\tau_1 (\delta'(\tau) G_{\psi\psi}(-\tau) + (q - 1) \delta(\tau) G_{bb}(-\tau)), \tag{1.6}
\]

after some simplification. Following [39] we can check that this notion of charges matches with the expectation

\[
\tilde{Q} = \frac{G_{\psi\psi}(0^+) + G_{\psi\psi}(0^-)}{2} + \text{constant}, \tag{1.7}
\]

\[
= \frac{Q}{N} + \text{constant}. \tag{1.8}
\]

To obtain this expression we defined the value of \( G_{\psi\psi}(0) \) as the average and we set \( \tau D(\tau)|_{\tau \to 0} \) to a constant to be determined below by consistency. The last step of the calculation is to perform the integral in the IR and match the UV answer above. This was already done in section 2.2.3 of [39] so we can simply quote the answer for the fermion in equation (4.1.48). The answer for the boson is simply given by a shift \( \mathcal{E} \to \mathcal{E} + i/2 \) since this removes extra minus signs that appear in fermion correlators. The final answer for the boson contribution is given in equation (4.1.49). This was of deriving the bosonic contribution has an ambiguity from the analytic continuation of the logarithm. We pick a sheet such that for \( \mathcal{E}_b = 0 \) the contribution to the charge vanishes as well. The total answer for the fermion charge is

\[
\frac{Q}{N} = q_f(\Delta, \mathcal{E}) + (q - 1)q_b(\Delta_b, \mathcal{E}_b). \tag{1.9}
\]

This fixes also the constant in the expression for \( \tilde{Q} \) above, demanding that when \( \mathcal{E} = 0 \) the
total fermion charge $Q$ must vanish.

Now we can generalize this to the models of section (4.2). In this case we have two $U(1)$ symmetries and therefore we expect two Luttinger-Ward relations relating $E_\chi$ and $E_\psi$ to $Q_\chi$ and $Q_\psi$. The two symmetries act on the correlators as

$$G_{\psi\psi}(\tau_1, \tau_2) \rightarrow e^{i\lambda(\tau_1)}e^{-i\lambda(\tau_2)}G_{\psi\psi}(\tau_1, \tau_2), \quad G_{\chi\chi}(\tau_1, \tau_2) \rightarrow G_{\chi\chi}(\tau_1, \tau_2),$$

$$G_{b_\psi b_\psi}(\tau_1, \tau_2) \rightarrow e^{i(q-2)\lambda(\tau_1)}e^{-i(q-2)\lambda(\tau_2)}G_{b_\psi b_\psi}(\tau_1, \tau_2), \quad G_{b_\chi b_\chi}(\tau_1, \tau_2) \rightarrow e^{i(q-1)\lambda(\tau_1)}e^{-i(q-1)\lambda(\tau_2)}G_{b_\chi b_\chi}(\tau_1, \tau_2),$$

and the other symmetry is

$$G_{\psi\psi}(\tau_1, \tau_2) \rightarrow G_{\psi\psi}(\tau_1, \tau_2), \quad G_{\chi\chi}(\tau_1, \tau_2) \rightarrow e^{i\lambda(\tau_1)}e^{-i\lambda(\tau_2)}G_{\chi\chi}(\tau_1, \tau_2),$$

$$G_{b_\psi b_\psi}(\tau_1, \tau_2) \rightarrow e^{i\lambda(\tau_1)}e^{-i\lambda(\tau_2)}G_{b_\psi b_\psi}(\tau_1, \tau_2), \quad G_{b_\chi b_\chi}(\tau_1, \tau_2) \rightarrow G_{b_\chi b_\chi}(\tau_1, \tau_2).$$

Now it is straightforward to generalize the previous derivation. In order to do this we introduce two bilocal currents $j(\tau_1, \tau_2)$ one for each symmetry and evaluate it in both the UV and IR. The final answer for the two charges is given in equations (4.2.63) and (4.2.64).

We have fixed ambiguities regarding the UV behavior of the boson two-point function by demanding that at zero spectral asymmetry the charge should vanish, similar to the case in the previous section.

.2 Kernels of the model with mutiple fermions in components

We can decompose the 2 by 2 super-kernel components wise. The fermionic ones give

$$(\delta G_{\psi b} \delta G_{\chi B}) = \begin{pmatrix} 2JG_{\psi\psi}(t_{14})G_{b_\psi b_\psi}(t_{32})G_{\chi\chi}(t_{34}) & 2JG_{\psi\psi}(t_{14})G_{b_\psi b_\psi}(t_{32})G_{\psi\psi}(t_{34}) \\ 2JG_{\chi\chi}(t_{14})G_{b_\chi b_\chi}(t_{32})G_{\psi\psi}(t_{34}) & 0 \end{pmatrix} \begin{pmatrix} \delta G_{\psi b} \\ \delta G_{\chi B} \end{pmatrix} $$

(2.1)
The bosonic components consist of two distinct sectors, the neutral sector and the charged sector. The charged sector can be further decomposed into the bosonic and fermionic parts

\[
\begin{pmatrix}
\delta G_{\psi\chi} \\
\delta G_{b\psi b\chi}
\end{pmatrix} = 2J \begin{pmatrix}
G_{\psi\psi}(t_{14})G_{\chi\chi}(t_{32})G_{b\psi b\psi}(t_{34}) & G_{\psi b\psi}(t_{14})G_{\chi\psi}(t_{32})G_{\psi\psi}(t_{34}) \\
-G_{b\psi b\psi}(t_{14})G_{b\psi b\chi}(t_{32})G_{\psi\psi}(t_{34}) & 0
\end{pmatrix} \begin{pmatrix}
\delta G_{\psi\chi} \\
\delta G_{b\psi b\chi}
\end{pmatrix}
\]

(2.2)

\[
\delta G_{\psi b\chi} = -2JG_{\psi\psi}(t_{14})G_{b\psi b\chi}(t_{32})G_{\psi\psi}(t_{34})\delta G_{\psi b\chi},
\]

(2.3)

\[
\delta G_{b\psi b\chi} = -2JG_{b\psi b\psi}(t_{14})G_{\chi\chi}(t_{32})G_{\psi\psi}(t_{34})\delta G_{b\psi b\chi},
\]

(2.4)

Finally the neutral sector for the bosonic correlators is given by

\[
\begin{pmatrix}
-2JG_{\psi b}(t_{14})G_{\psi b}(t_{32})G_{b\psi b}\chi(t_{34}) \\
-2JG_{\psi b}(t_{14})G_{b\psi b}(t_{32})G_{\chi\chi}(t_{34}) \\
2JG_{b\psi b}(t_{14})G_{b\psi b}(t_{32})G_{\psi\psi}(t_{34}) \\
2JG_{b\psi b}(t_{14})G_{b\psi b}(t_{32})G_{\psi\psi}(t_{34})
\end{pmatrix}
\]

(2.5)
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