Lectures on Cosmology

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1 Homogeneous and Isotropic Space-Times

In these lectures we will focus on one set of solutions to the Einstein Field Equations which apply in a very particular regime. We will solve them for the whole Universe under the assumption that it is homogeneous and isotropic.

For many centuries we have grown to believe that we don't live in a special place, that we are not at the center of the Universe. And, oddly enough, this point of view allows us to make some far reaching assumptions. So for example, if we are insignificant and, furthermore, everywhere is insignificant, then we can assume that at any given time, the Universe looks the same everywhere. In fact we can take that statement to an extreme and assume that at any given time, the Universe looks *exactly* the same at every single point in space. Such a space-time is dubbed to be *homogeneous*.

There is another assumption that takes into account the extreme regularity of the Universe and that is the fact that, at any given point in space, the Universe looks very much the same in whatever direction we look. Again such an assumption can be taken to an extreme so that at any point, the Universe look *exactly* the same, whatever direction one looks. Such a space time is dubbed to be *isotropic*.

Homogeneity and isotropy are distinct yet inter-related concepts. For example a universe which is isotropic will be homogeneous while a universe that is homogeneous *may not* be isotropic. A universe which is only isotropic around one point is not homogeneous. A universe that is both homogeneous and isotropic is said to satisfy the *Cosmological Principle*. It is believed that our Universe satisfies the Cosmological Principle.

Homogeneity severely restrict the metrics that we are allowed to work with in the Einstein field equation. For a start, they must be independent of space, and solely functions of time. Furthermore, we must restrict ourselves to spaces of constant curvature of which there are only three: a flat euclidean space, a positively curved space and a negatively curved space. We will look at curved spaces in a later lecture and will restrict ourselves to a flat geometry here.

The metric for a flat Universe takes the following form:

$$ds^2 = -c^2dt^2 + a^2(t)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2]$$

We call $a(t)$ the scale factor and t is normally called *cosmic* time or *physical* time. The energy momentum tensor must also satisfy homogeneity and isotropy. If we consider a perfect fluid, we restrict ourselves to

$$T^{\alpha\beta} = (\rho + \frac{P}{c^2})U^\alpha U^\beta + Pg^{\alpha\beta}$$

with $U^\alpha = (c, 0, 0, 0)$ and ρ and P are simply functions of time. Note that both the metric and the energy-momentum tensor are diagonal. So

$$
\begin{array}{ll}
g_{00} = -1 & g_{ij} = a^2(t)\delta_{ij} \\
T_{00} = \rho c^2 & T_{ij} = a^2 P \delta_{ij}
\end{array}
$$

As we shall see, with this metric and energy-momentum tensor, the Einstein field equations are greatly simplified. We must first calculate the connection coefficients. We have that the only non-vanishing elements are (and from now one we will use $\dot{} = \frac{d}{dt}$:

$$
\begin{aligned}
\Gamma^0{}_{ij} &= \frac{1}{c} a\dot{a}\delta_{ij} \\
\Gamma^i{}_{0j} &= \frac{1}{c}\frac{\dot{a}}{a}\delta^i_j
\end{aligned}
$$

and the resulting Ricci tensor is

$$
\begin{aligned}
R_{00} &= -\frac{3}{c^2}\frac{\ddot{a}}{a} \\
R_{0i} &= 0 \\
R_{ij} &= \frac{1}{c^2}(a\ddot{a} + 2\dot{a}^2)\delta_{ij}
\end{aligned}
$$

Again, the Ricci tensor is diagonal. We can calculate the Ricci scalar:

$$
R = -R_{00} + \frac{1}{a^2}R_{ii} = \frac{1}{c^2}\left[6\frac{\ddot{a}}{a} + 6\left(\frac{\dot{a}}{a}\right)^2\right]
$$

to find the two Einstein Field equations:

$$
\begin{aligned}
G_{00} = R_{00} - \frac{1}{2}Rg_{00} = \frac{8\pi G}{c^4}T_{00} \;&\Leftrightarrow\; 3\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G\rho \\
G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij} = \frac{8\pi G}{c^4}T_{ij} \;&\Leftrightarrow\; -2a\ddot{a} - \dot{a}^2 = \frac{8\pi G}{c^2}a^2 P
\end{aligned}
$$

We can use the first equation to simplify the 2^{nd} equation to

$$
3\frac{\ddot{a}}{a} = -4\pi G(\rho + 3\frac{P}{c^2})
$$

These two equations can be solved to find how the scale factor, $a(t)$, evolves as a function of time. The first equation is often known as the *Friedmann-Robertson-Walker* equation or FRW equation and the metric is one of the three FRW metrics. The latter equation in $\ddot{a}$ is known as the *Raychauduri* equation.

Both of the evolution equations we have found are sourced by ρ and P. These quantities satisfy a conservation equation that arises from

$$
\nabla_\alpha T^{\alpha\beta} = 0
$$

and in the homogeneous and isotropic case becomes

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + \frac{P}{c^2}) = 0$$

It turns out that the FRW equation, the Raychauduri equation and the energy-momentum conservation equation are not independent. It is a straightforward exercise to show that you can obtain one from the other two. We are therefore left with two equations for three unknowns.

One has to decide what kind of energy we are considering and in a later lecture we will consider a variety of possibilities. But for now, we can hint at a substantial simplification. If we assume that the system satisfies an equation of state, so $P = P(\rho)$ and, furthermore that it is a *polytropic fluid* we have that

$$P = w\rho c^2 \tag{1}$$

where w is a constant, the *equation of state* of the system.

2 Properties of a Friedmann Universe I

We can now explore the properties of these evolving Universes. Let us first do something very simple. Let us pick two objects (galaxies for example) that lie at a given distance from each other. At time t_1 they are at a distance r_1 while at a time t_2, they are at a distance r_2. We have that during that time interval, the change between r_1 and r_2 is given by

$$\frac{r_2}{r_1} = \frac{a(t_2)}{a(t_1)}$$

and, because of the cosmological principle, this is true *whatever* two points we would have chosen. It then makes to sense to parametrize the distance between the two points as

$$r(t) = a(t)x$$

where x is completely independent of t. We can see that we have already stumbled upon x when we wrote down the metric for a homogeneous and isotropic space time. It is the set of coordinates (x^1, x^2, x^3) that remain unchanged during the evolution of the Universe. We known that the real, *physical* coordinates are multiplied by $a(t)$ but (x^1, x^2, x^3) are time independent and are known as *conformal* coordinates. We can work out how quickly the two objects we considered are moving away from each other. We have that their relative velocity is given by

$$v = \dot{r} = \dot{a}x = \frac{\dot{a}}{a}ax = \frac{\dot{a}}{a}r \equiv Hr$$

In other words, the recession speed between two objects is proportional to the distance between them. This equality applied today (at t_0) is

$$v = H_0 r$$

and is known as *Hubble's Law* where H_0 is the Hubble constant and is given by $H_0 = 100h$ km s^{-1} Mpc^{-1} and h is a dimensionless constant which is approximately $h \simeq 0.7$.

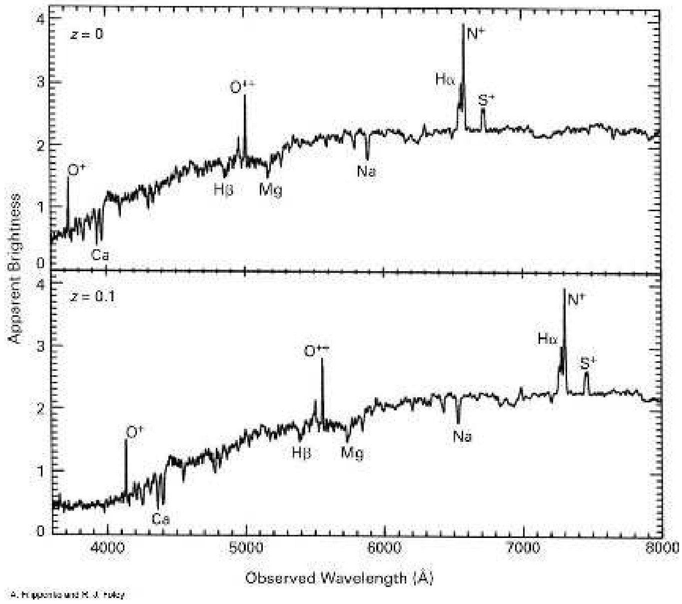

Figure 1: A set of spectra measured in laboratory (top panel) on a distant galaxy (bottom panel)

How can we measure velocities in an expanding universe? Consider a photon with wavelength λ being emitted at one point and observed at some other point. We have that the Doppler shift is given by

$$\lambda' \simeq \lambda(1 + \frac{v}{c})$$

We can rewrite it in a differential form

$$\frac{d\lambda}{\lambda} \simeq \frac{dv}{c} = \frac{\dot{a}}{a}\frac{dr}{c} = \frac{\dot{a}}{a}dt = \frac{da}{a}$$

and integrate to find $\lambda \propto a$. We therefore have that wave lengths are stretched with the expansion of the Universe. It is convenient to define the factor by which the wavelength is stretched by

$$z = \frac{\lambda_r - \lambda_e}{\lambda_e} \to 1 + z \equiv \frac{a_0}{a}$$

where a_0 is the scale factor today (throughout these lecture notes we will choose a convention in which $a_0 = 1$). We call z the *redshift*.

For example, if you look at Figure 1 you can see the spectra measured from a galaxy; a few lines are clearly visible and identifiable. Measured in the laboratory on Earth (top panel), these lines will have a specific set of wavelengths but measured in a specific, distant, galaxy (bottom panel) the lines will be shifted to longer wavelengths. Hence a measurement of the redshift (or blueshift), i.e. a measurement of the Doppler shift, will be a direct measurement of the velocity of the galaxy.

The American astronomer, Edwin Hubble measured the distances to a number of distant galaxies and measured their recession velocities. The data he had was patchy, as you can see

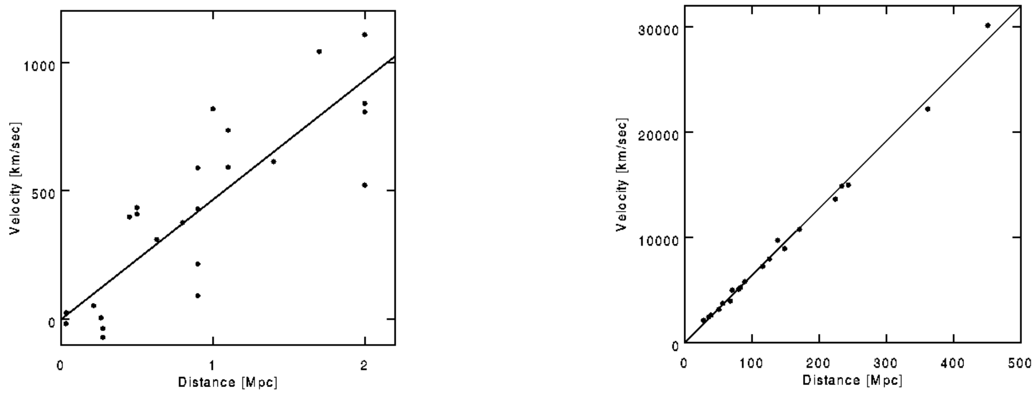

Figure 2: The recession velocity of galaxies, Hubble's data circa 1929 (left) and SN data circa 1995 (right)

from Figure 2, but he was able to discern a a pattern: most of the galaxies are moving away from us *and* the further away they are, the faster they are moving. With more modern data, this phenomenon is striking, as you can see in the Figure 2. The data is neatly fit by a law of the form

$$v = H_0 r$$

where H_0 is a constant (known as Hubble's constant). Current measurements of this constant give us $H_0 = 67$ km s^{-1} Mpc^{-1}.

3 Energy, Pressure and the History of the Universe

We can now solve the FRW equations for a range of different behaviours. In the final few lectures we will look, in some detail, at the nature of matter and energy in an expanding Universe but for now, we will restrict ourselves to describing them in terms of their equation of state in the form given in equation 1, $P = w\rho c^2$.

Let us start off with the case of *non-relativistic* matter. A notable example is that of massive particles whose energy is dominated by the rest energy of each individual particle. This kind of matter is sometimes simply called *matter* or *dust*. We can guess what the evolution of the mass density should be. The energy in a volume V is given by $E = Mc^2$ so $\rho c^2 = E/V$ where ρ is the mass density. But in an evolving Universe we have $V \propto a^3$ so $\rho \propto 1/a^3$. Alternatively, note that $P \simeq nk_BT \ll nMc^2 \simeq \rho c^2$ so $P \simeq 0$. Hence, using the conservation of energy equations:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho = \frac{1}{a^3}\frac{d}{dt}(\rho a^3) = 0$$

and solving this equation we find $\rho \propto a^{-3}$. We can now solve the FRW equation (taking $\rho(a = 1)\rho_0)$:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{\rho_0}{a^3}$$

$$a^{1/2}\dot{a} \;=\; \left(\frac{8\pi G\rho_0}{3}\right)^{1/2}$$

to find $a \propto t^{2/3}$. If $a(t_0) = 1$, where t_0 is the time today, we have $a = (t/t_0)^{2/3}$ You will notice a few things. First of all, at $t = 0$ we have $a = 0$ i.e. there is an initial *singularity* known as the *Big Bang*. Furthermore we have that $v = \frac{\dot{a}}{a}r = \frac{2}{3t_0}r$. i.e. by measuring Hubble's law we measure the age of the Universe. And finally we have that $\ddot{a} < 0$ i.e. the Universe is *decelerating*.

The case of *Relativistic Matter* encompasses particles which are massless like photons or neutrinos. Recall that their energy is given by $E = h\nu = h2\pi/\lambda$ where ν is the frequency and λ is the wavelength. As we saw in Section 2, wavelengths are redshifted, i.e. $\lambda \propto a$ and hence the energy of an individual particle will evolve as $E \propto 1/a$. Once again, the mass density is given by $\rho c^2 = E/V \propto 1/(V\lambda) \propto 1/(a^3 a) = 1/a^4$. So the energy density of radiation decreases far more quickly than that of dust. We can go another route, using the equation of state and conservation of energy. We have that for radiation $P = \rho c^2/3$, so

$$\dot{\rho} + 4\frac{\dot{a}}{a}\rho = \frac{1}{a^4}\frac{d}{dt}(\rho a^4) = 0$$

which can be solved to give $\rho \propto a^{-4}$. We can now solve the FRW equations.

$$\left(\frac{\dot{a}}{a}\right)^2 \;=\; \frac{8\pi G}{3}\frac{\rho_0}{a^4}$$

$$a\dot{a} \;=\; \left(\frac{8\pi G\rho_0}{3}\right)^{1/2}$$

to find $a \propto t^{1/2}$ or $a = (t/t_0)^{1/2}$ Once again, the universe is decelerating but now $H_0 = 1/(2t_0)$. Note that there is a different relation between H_0 and t_0 so if we are to infer the age of the Universe from the expansion rate, we need to know what it contains.

It is straightforward to consider a general w. Energy-momentum conservation gives us

$$\dot{\rho} + 3(1+w)\frac{\dot{a}}{a}\rho = \frac{1}{a^{3(1+w)}}\frac{d}{dt}(\rho a^{3(1+w)}) = 0$$

Note that as w gets smaller and more negative, ρ decays more slowly. We can solve the FRW equations

$$\left(\frac{\dot{a}}{a}\right)^2 \;=\; \frac{8\pi G}{3}\frac{\rho_0}{a^{3(1+w)}}$$

$$a^{(1+3w)/2}\dot{a} \;=\; \left(\frac{8\pi G\rho_0}{3}\right)^{1/2}$$

to find $a = (t/t_0)^{2/3(1+w)}$ which is valid if $w > -1$. For $w < -1/3$ the expansion rate is *accelerating*, not decelerating. For the special case of $w = -1/3$ we have $a \propto t$.

Finally, we should consider the very special case of a *Cosmological Constant*. Such odd situation arises in the extreme case of $P = -\rho c^2$. You may find that such an equation of state is obeyed by vaccum fluctuations of matter. Such type of matter can be described by the Λ we

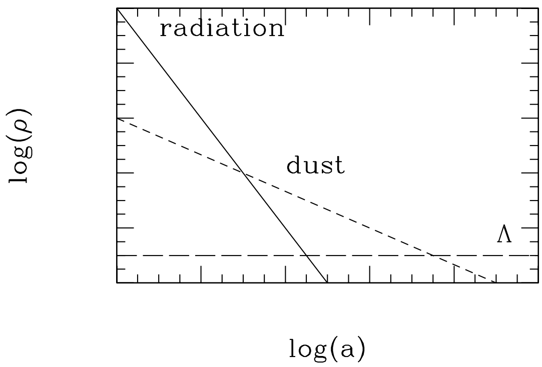

Figure 3: The energy density of radiation, matter and the cosmological constant as a function of time

found in the Einstein Field Equations. The solutions are straightforward: ρ is constant, $\frac{\dot{a}}{a}$ is constant and $a \propto \exp(Ht)$.

Throughout this section, we have considered one type of matter at a time but it would make more sense to consider a mix. For example we know that there are photons *and* protons in the Universe so in the very least we need to include both types of energy density in the FRW equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\frac{\rho_{M0}}{a^3} + \frac{\rho_{R0}}{a^4}\right)$$

In fact, the current picture of the universe involves all three types of matter/energy we considered in this section and, depending on their evolution as a function of a, they will dominate the dynamics of the Universe at different times. In Figure 3 we plot the energy densities as a function of scale factor and we can clearly see the three stages in the Universe's evolution: a *radiation era*, followed by a *matter era* ending up with a *cosmological constant era* more commonly known as a Λ *era*.

4 Geometry and Destiny

Until now we have restricted ourselves to a flat Universe with Euclidean geometry. Before we move away from such spaces let us revisit the metric. We have

$$ds^2 = -c^2dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$$

Let us transform to spherical polar coordinates

$$\begin{aligned} x &= r\cos\phi\sin\theta \\ y &= r\sin\phi\sin\theta \\ z &= r\cos\theta \end{aligned}$$

and rewrite the metric

$$ds^2 = -c^2dt^2 + a^2(t)(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2)$$

We could in principle work out the FRW and Raychauduri equations in this coordinate system.

Let us now consider a 3 dimensional surface that is positively curved. In other words, it is the surface of a 3 dimensional hypersphere in a fictitious space with 4 dimensions. The equation for the surface of a sphere in this 4 dimensional space, with coordinates (X, Y, Z, W), is

$$X^2 + Y^2 + Z^2 + W^2 = R^2$$

Now in the same way that we can construct spherical coordinates in three dimensions, we can build hyperspherical coordinates in 4 dimensions:

$$\begin{aligned} X &= R\sin\chi\sin\theta\cos\phi \\ Y &= R\sin\chi\sin\theta\sin\phi \\ Z &= R\sin\chi\cos\theta \\ W &= R\cos\chi \end{aligned}$$

We can now work out the line element on the surface of this hyper-sphere

$$ds^2 = dX^2 + dY^2 + dZ^2 + dW^2 = R^2\left[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

Note how different it is from the flat geometry. If we transform $R\sin\chi$ into r for it all to agree we have that

$$d\chi^2 = \frac{dr^2}{R^2 - r^2}$$

We can now repeat this exercise for 3-D surface with negative curvature- a hyper-hyperboloide so to speak. In our fictitious 4-D space (not to be confused with space time), we have that the surface is defined by

$$X^2 + Y^2 + Z^2 - W^2 = -R^2$$

Let us now change to a good coordinate system for that surface:

$$\begin{aligned} X &= R\sinh\chi\sin\theta\cos\phi \\ Y &= R\sinh\chi\sin\theta\sin\phi \\ Z &= R\sinh\chi\cos\theta \\ W &= R\cosh\chi \end{aligned}$$

The line element on that surface will now be

$$ds^2 = dX^2 + dY^2 + dZ^2 + dW^2 = R^2\left[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

We can replace $R\sinh\chi$ by r to get

$$d\chi^2 = \frac{dr^2}{R^2 + r^2}$$

We can clearly write all three space time metrics (flat, hyperspherical, hyper-hyperbolic) in a unified way. If we take $r = R\sin\chi$ for the positively curved space and $r = R\sinh\chi$ for the negatively curved space we have

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{d^2r}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{2}$$

where k is positive, zero or negative for spherical, flat or hyperbolic geometries, and $|k| = 1/R^2$.

We can now repeat the calculation we undertook for a flat geometry and find the connection coefficients, Ricci tensor and scalar and the evolution equations. Take the metric $g_{\alpha\beta} = diag(-1, \frac{a^2}{1-kr^2}, a^2r^2, a^2r^2\sin\theta^2)$ and note that for this choice of coordinates, the i and j labels now run over r, θ and ϕ. We find that the connection coefficients are:

$$\begin{aligned}\Gamma^0_{ij} &= \frac{1}{c}a\dot{a}\tilde{g}_{ij}\\ \Gamma^i_{0j} &= \frac{1}{c}\frac{\dot{a}}{a}\delta^i_j\\ \Gamma^i_{jk} &= \tilde{\Gamma}^i_{jk}\end{aligned}$$

where $\tilde{g}_{ij}$ and $\tilde{\Gamma}$ are the metric and connection coefficients of the conformal 3-space (that is of the 3-space with the conformal factor, a, divided out):

$$\begin{aligned}\tilde{\Gamma}^r{}_{rr} &= \frac{kr}{1-kr^2}\\ \tilde{\Gamma}^r{}_{\theta\theta} &= -r(1-kr^2)\\ \tilde{\Gamma}^r{}_{\phi\phi} &= -(1-kr^2)r\sin^2(\theta)\\ \tilde{\Gamma}^\theta{}_{\theta r} &= \frac{1}{r}\\ \tilde{\Gamma}^\theta{}_{\phi\phi} &= \frac{-\sin(2\theta)}{2}\\ \tilde{\Gamma}^\phi{}_{\phi r} &= \frac{1}{r}\\ \tilde{\Gamma}^\phi{}_{\theta\phi} &= \frac{1}{\tan(\theta)}\end{aligned}$$

The Ricci tensor and scalar can be combined to form the Einstein tensor

$$\begin{aligned}G_{00} &= 3\frac{\dot{a}^2 + kc^2}{c^2a^2}\\ G_{ij} &= -\frac{2a\ddot{a} + \dot{a}^2 + kc^2}{c^2}\tilde{g}_{ij}\end{aligned} \tag{3}$$

while the energy-momentum tensor is

$$\begin{aligned}T_{00} &= \rho c^2\\ T_{ij} &= a^2P\tilde{g}_{ij}\end{aligned}$$

Combining them gives us the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

while the Raychauduri equations remains as

$$3\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3\frac{P}{c^2})$$

Let us now explore the consequences of the overall geometry of the Universe, i.e. the term proportional to k in the FRW equations: For simplicity, let us consider a dust filled universe. We can see that the term proportional to k will only be important at late times, when it dominates over the energy density of dust. In other words, in the universe we can say that *curvature dominates* at late times. Let us now consider the two possibilities. First of all, let us take $k < 0$. We then have that

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{|k|c^2}{a^2}$$

When the curvature dominates we have that

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{|k|c^2}{a^2}$$

so $a \propto t$. In this case, the scale factor grows at the speed of light. We can also consider $k > 0$. From the FRW equations we see that there is a point, when $\frac{8\pi G}{3}\rho = \frac{kc^2}{a^2}$ and therefore $\dot{a} = 0$ when the Universe stops expanding. At this point the Universe starts contracting and evolves to a *Big Crunch*. Clearly geometry is intimately tied to destiny. If we know the geometry of the Universe we know its future.

There is another way we can fathom the future of the Universe. If $k = 0$, there is a strict relationship between $H = \frac{\dot{a}}{a}$ and ρ. Indeed from the FRW equation we have

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \rightarrow \rho = \rho_c \equiv \frac{3H^2}{8\pi G}$$

We call ρ_c the *critical density*. It is a function of a. If we take $H_0 = 100h\text{Km s}^{-1}\text{ Mpc}^{-1}$, we have that

$$\rho_c = 1.9 \times 10^{-26}h^2\text{kgm}^{-3}$$

which corresponds to a few atoms of Hydrogen per cubic meter. Compare this with the density of water which is 10^3 kg m^{-3}. Now let us take another look at the FRW equation and rewrite it as

$$\frac{1}{2}\dot{a}^2 - \frac{4\pi G}{3}\rho a^2 = -\frac{1}{2}kc^2$$

which has the form $E_{tot} = U + K$ and we equate E_{tot} to $-kc^2$ so that K is the kinetic energy, U is the gravitational energy. We see that if $\rho = \rho_c$, it corresponds to the total energy of the system being 0, i.e. kinetic and gravitational energy balance themselves out perfectly. Let us look at the case of nonzero k.

$k < 0$ $\rho < \rho_c$ and therefore total energy is positive, kinetic energy wins out and the Universe expands at a constant speed.

$k > 0$ $\rho > \rho_c$ and the total energy is negative, gravitational energy wins out and the Universe recollapses.

We recover an important underlying principle behind all this, the geometry is related to the energy density.

It is convenient to define a more compact notation. The *fractional energy density* or *density parameter*. We define

$$\Omega \equiv \frac{\rho}{\rho_c}$$

It will be a function of a and we normally express its value today as Ω_0. If there are various contributions to the energy density, we can define the fractional energy densities of each one of these contributions. For example

$$\Omega_R \equiv \frac{\rho_R}{\rho_c} \qquad \Omega_M \equiv \frac{\rho_M}{\rho_c} \qquad \cdots$$

It is convenient to define two additional Ωs:

$$\begin{aligned} \Omega_\Lambda &\equiv \frac{\Lambda}{3H^2} \\ \Omega_k &\equiv -\frac{kc^2}{a^2H^2} \end{aligned}$$

and we have Ω:

$$\Omega = \Omega_R + \Omega_M + \Omega_\Lambda$$

We now have

$\Omega < 1$: $\rho < \rho_c$, $k < 0$, Universe is open (hyperbolic)

$\Omega = 1$: $\rho = \rho_c$, $k = 0$, Universe is flat (Euclidean)

$\Omega > 1$: $\rho > \rho_c$, $k > 0$, Universe is closed (spherical)

If we divide the FRW equation through by ρ_c we find that it can be rewritten as

$$H^2(a) = H_0^2 \left[\frac{\Omega_{M0}}{a^3} + \frac{\Omega_{R0}}{a^4} + \frac{\Omega_{K0}}{a^2} + \Omega_\Lambda \right] \tag{4}$$

where the subscript "0" indicates that these quantities are evaluated at t_0. We will normally drop the subscript when referring to the various Ωs evaluated today. When we refer to the Ωs at different times, we will explicitely say so or add an argument (for example $\Omega_M(a)$ or $\Omega_M(z)$).

How does Ω evolve? Without loss of generality, let us consider a Universe with dust, take the FRW equations and divide by H^2 to obtain

$$\Omega - 1 = \frac{kc^2}{a^2H^2} \propto kt^{2/3}$$

I.e., if $\Omega \neq 1$, it is unstable and driven away from 1. The same is true in a radiation dominated universe and for any *decelerating* Universe: $\Omega = 1$ is an unstable fixed point and, as we saw above, curvature dominates at late times.

5 Properties of a Friedmann Universe I

Let us revisit the properties of a FRW universe, now that we know a bit more about the the evolution of the scale factor. Distances play an important role if we are to map out its behaviour in detail. We have already been exposed to Hubble's law

$$v = H_0 d$$

from which we can extract Hubble's constant. From Hubble's constant we can define a *Hubble time*

$$t_H = \frac{1}{H_0} = 9.78 \times 10^9 \ h^{-1} \text{ yr}$$

and the *Hubble distance*

$$D_H = \frac{c}{H_0} = 3000 \ h^{-1} \text{ Mpc}$$

These quantities set the scale of the Universe and give us a rough idea of how old it is and how far we can see. They are only rough estimates and to get a firmer idea of distances and ages, we need to work with the metric and FRW equations more carefully.

To actually figure out how far we can see, we need to work out how far a light ray travels over a given period of time. To be specific, what is the distance, D_M to a galaxy that emitted a light ray at time t, which reaches us today? Let us look at the expression for the metric used in equation 2 for a light ray. We have that

$$\frac{dr^2}{1 - kr^2} = \frac{c^2 dt^2}{a^2(t)} \tag{5}$$

The time integral gives us the *comoving distance*:

$$D_C = c \int_t^{t_0} \frac{dt'}{a(t')}$$

From equation 4 we have that $-k = \Omega_k / D_H^2$. Performing the radial integral (and assuming the observer is at $r = 0$ we have

$$\int_0^{D_M} \frac{dr}{\sqrt{1 - kr^2}} = \begin{cases} \frac{D_H}{\sqrt{\Omega_k}} \sinh^{-1}[\sqrt{\Omega_k} D_M / D_H] & \text{for } \Omega_k > 0 \\ D_M & \text{for } \Omega_k = 0 \\ \frac{D_H}{\sqrt{|\Omega_k|}} \sin^{-1}[\sqrt{|\Omega_k|} D_M / D_H] & \text{for } \Omega_k < 0 \end{cases}$$

so we find an expression for the *proper motion distance* (also known as the *transverse comoving distance*, D_M in terms of the comoving distance)

$$D_M = \begin{cases} \frac{D_H}{\sqrt{\Omega_k}} \sinh[\sqrt{\Omega_k} D_C / D_H] & \text{for } \Omega_k > 0 \\ D_C & \text{for } \Omega_k = 0 \\ \frac{D_H}{\sqrt{|\Omega_k|}} \sin[\sqrt{|\Omega_k|} D_C / D_H] & \text{for } \Omega_k < 0 \end{cases}$$

Suppose now we we look at an object of a finite size which is transverse to our line of sight and lies at a certain distance from us. If we divide the physical transverse size of the object by the angle that object subtends in the sky (the angular size of the object) we obtain the *angular diameter distance*:

$$D_A = \frac{D_M}{1+z}$$

Hence, if we know that size of an object and its redshift we can work out, for a given Universe, D_A.

Alternatively, we may know the brightness or luminosity of an object at a given distance. We know that the flux of that object at a distance D_L is given by

$$F = \frac{L}{4\pi D_L^2}$$

D_L is aptly known as the *luminosity distance* and is related to other distances through:

$$D_L = (1+z)D_M = (1+z)^2 D_A$$

It turns out that, in astronomy, one often works with a logarithmic scale, i.e. with *magnitudes*. One can define the *distance modulus*:

$$DM \equiv 5\log\left(\frac{D_L}{10\ \mathrm{pc}}\right)$$

and it can be measured from the *apparent magnitude* m (related to the flux at the observer) and the *absolute magnitude* M (what it would be if the observer was at 10 pc from the source) through

$$m = M + DM$$

We now have a plethora of distances which can be deployed in a range of different observations. They clearly depend on the universe we are considering, i.e. on the values of H_0, and the various Ωs. While Ω_k will dictate the geometry, D_c will depend on how the Universe evolves. It is useful to rewrite D_c in a few different ways. It is useful to use the FRW in the form presented in equation 4. We can transform the time integral in D_c to an integral in a:

$$D_C = \int_t^{t_0} \frac{cdt'}{a(t')} = c\int_a^1 \frac{da}{a^2 H(a)} = D_H \int_a^1 \frac{da}{a^2\sqrt{\Omega_M/a^3 + \Omega_R/a^4 + \Omega_k/a^2 + \Omega_\Lambda}}$$

An interesting question is how far has light travelled, from the big bang until now? This is known as the *particle horizon*, r_P and a naive estimate would be $r_P \simeq ct_0$ but that doesn't take into account the expansion of space time. The correct expression is given above and it is

$$r_P = D_C(0)$$

where the argument implies that it is evaluated from $t = 0$ to $t = t_0$. Applying it now to the simple case of a dust filled, flat Universe. We have that

$$r_P = 3ct_0$$

Unsurprisingly, the expansion leads to an extra factor.

We could ask a different question: how far can light travel from now until the infinite future, i.e. how much will we ever see of the current Universe. Known as the *event horizon* it is by the integral of equation 5 from t_0 until ∞. For example in a flat Universe we have

$$r_E = \int_{t_0}^{\infty} \frac{cdt'}{a(t')}$$

For a dust or radiation dominated universe we have that $r_E = \infty$ but this is not so for a universe dominated by a cosmological constant.

We have been focusing on distances but we can also improve our estimate of ages. We defined the Hubble time above and that is a rough estimate of the age of the Universe. To do better we need to resort to the FRW equations again, as above we have that $\dot{a} = aH$ so

$$dt = \frac{da}{aH} \to \int_0^{t_0} dt = \int_0^1 \frac{da}{aH} = t_0$$

which, combined with equation 4 gives us

$$t_0 = \frac{1}{H_0} \int_0^1 \frac{da}{a\sqrt{\Omega_M/a^3 + \Omega_R/a^4 + \Omega_k/a^2 + \Omega_\Lambda}}$$

We can use the above equation quite easily. For a flat, dust dominated Universe we find $t_0 = 2/(3H_0)$. If we now include a cosmological constant as well, we find

$$t_0 = H_0^{-1} \int_0^1 \frac{da}{a\sqrt{\Omega_M/a^3 + \Omega_\Lambda}}$$

At $\Omega_\Lambda = 0$ we simply retrieve the matter dominated result, but the larger Ω_Λ is, the older the Universe. To understand why, recall the Raychauduri equation for this Universe:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho + \frac{\Lambda}{3}$$

Divide by H_0^2 and we have that the *deceleration parameter*

$$q_0 \equiv -\frac{a(t_0)\ddot{a}(t_0)}{\dot{a}^2(t_0)} = \frac{1}{2}\Omega_M - \Omega_\Lambda$$

If $\Omega_M + \Omega_\Lambda = 1$ then $q_0 = \frac{3}{2}\Omega_M - 1$. If $\Omega_M < \frac{2}{3}$ we have $q_0 < 0$ and the Universe is accelerating. This lets us understand why the Universe is older. Take a $\Omega_\Lambda = 0$ and a $\Omega_\Lambda > 0$ which both have the *same* expansion rate today. The latter is accelerating which means it was expanding more slowly in the past than the former. This means it must have taken longer to reach its current speed and hence is older. Furthermore we can see that our inference about the Universe depends on our knowledge of the various Ωs. In other words, if we want to measure the age of the Universe we must also measure the density in its various components.

Finally, let us revisit Hubble's law. We worked out the relationship between velocities and distance for two objects which were very close to each other. If we want to consider objects

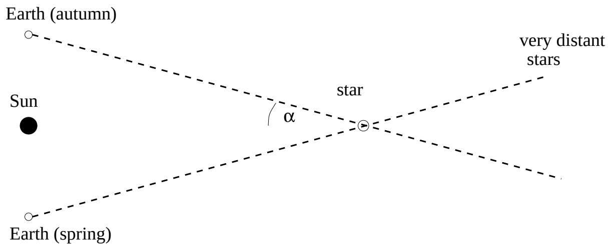

Figure 4: The motion of the Earth around the Sun supplies us with a long baseline for parallax measurments.

which are further apart (not too distant galaxies) we can Taylor expand the scale factor today, we find that

$$a(t) = a(t_0) + \dot{a}(t_0)[t - t_0] + \frac{1}{2}\ddot{a}(t_0)[t - t_0]^2 + \cdots$$

Assume that the distance to the emitter at time t is roughly given by $d =\simeq c(t_0 - t)$ we can rewrite it as

$$(1+z)^{-1} = 1 - H_0\frac{d}{c} - \frac{q_0 H_0^2}{2}\left(\frac{d}{c}\right)^2 + \cdots$$

For $q_0 = 0$ and small z we recover the Hubble law, $cz = H_0 d$. As we go to higher redshift, this is manifestly not good enough.

6 The Cosmological Distance ladder

Given our model of a range of possible universes, we would like to pin down which set of *cosmological parameters* (like t_0, H_0, Ω_M, $\cdots$) correspond to our Universe. We can ask questions like: what is the age of the Universe, is it accelerating or decelerating, what is its density and geometry? Interestingly enough, all these questions must be answered together and to do so we need to go out, observe and measure.

The first step is to map out the Universe and measure distances and redshifts accurately. By far the easiest quantity to measure is the redshift. By looking at the shift in the spectra of known elements it is possible to infer the recession velocity of the galaxy directly. Measuring distances is much harder. The most direct method is to use parallax to measure the distance to a star. Let us remember what you do here. Imagine that you look at an object in the sky. It can be described in terms of two angles. It has a position on the celestial sphere. Now imagine that we move a distance $2d$ from where we were. The object may move an angle θ from where it was. The angle that it has moved will be related to the distance D and displacement d. If we say $\theta = 2\alpha$ then we have $\tan\alpha = \frac{d}{D}$ If α is small then we can use the small angle approximation to get

$$\alpha = \frac{d}{D}$$

The motion of the earth around the sun gives us a very good baseline with which to measure distance. The distance from the earth to the sun is 1 AU so we have that $D = \frac{1}{\alpha}$ where α is measured in arcseconds. D is then given in *parsecs.* One parsec corresponds to 206,265 AU or 3.09×10^{13}km. This is a tremendous distance, $1pc \sim 3.26$ light years. All stars have parallax angles less than one arcsecond. The closest star, Proxima Centauri, has a distance of 1.3pc. In 1989 a satellite was launched called Hipparcos to measure the distances to 118,000 stars with an accuracy of 0.001 arcseconds. This corresponds to distances of hundreds of parsecs. This may seem far but it isn't. The sun is 8kpc away from the centre of the galaxy.

We would like to be able to look further. The basic tool for doing this is to take an object of known brightness and see how bright it looks. Take a star with a given luminosity L. The luminosity is the amount of light it pumps out per second. How bright will it look from where we stand? We can think of standing on a point of a sphere of radius D centred on the star. The brightness will be $B = \frac{L}{4\pi D^2}$ The further away it is the dimmer it will look. If we know the luminosity of a star and we measure its brightness, then we will know how far away it is.

How can we do that in practice? Stars have varying luminosities and are very different. Is there any way in which we can use information about a star's structure to work out it's luminosity? Let us start by looking at the colours of stars. Different stars will emit different spectra. Some will look redder, others more yellow, while others will be blue. Their colours (or spectra) are intimately tied to their temperature. Remember a black body what black body looks like. Its spectrum peaks at a certain value which is given by its temperature. For example, the Sun is yellow-white, has a temperature of 5800 K. The star Bellatra is blue and has a temperature of 21,500 K. Betelgeuse is red and has a temperature of 3500 K. Now we might think that we have it made.

The luminosity must be related to the temperature somehow. If we assume that it is black body, the energy flux is $F = \sigma T^4$ where σ is the Stefan-Boltzmann constant $\sigma = 5.6\times 10^{-8} W m^{-2} K^{-4}$. So luminosity is simply the surface of the star times its flux $L = 4\pi R^2 \sigma T^4$. There is indeed a very tight connection *but* stars can have different radii. For example main sequence stars have one type of radius while red giants have much larger radii. We can look at the H.R. diagram and find stars with the same temperature which have very different luminosities. However if we can identify what type of stars they are then we can, given their colours, read off their luminosities.

Suppose we look at the spectra of two stars, A and B, and we identify some spectral lines. These correspond to the same absorption/emission lines but in A they're narrower than in B. What leads to the thickness of the lines? If there are random velocities, they will Doppler shift the line. The larger the spread in velocities, the more shifts there will be. But clearly for there to be a larger spread, they have to be closer to the core of the star i.e. the radius has to be smaller. I.e. broader lines imply smaller R. So by reading off the thickness of the lines we can pinpoint what type of stars they are and then from their colours we can infer their luminosity. For example: Sun has $T \sim 5800K$. It is a main sequence star with a luminosity of $1\,\mathrm{L}_\odot$. Aldebaran is a giant star which, even though it is cooler, $T \sim 4000K$, has a luminosity of $370\,\mathrm{L}_\odot$. This method, known as *spectroscopic parallax* can be used to go out to 10kpc.

How can we move out beyond 10kpc? There are some stars which have a very useful property. Their brightness varies with time and the longer their variation, the larger their luminosity. These stars known as *Cepheid* stars are interesting because they have a) periods of days (which means their variations can be easily observed) and b) are very luminous with luminosites of about $100 - 1000\,\mathrm{L}_\odot$ and therefore they can be seen at great distances. It was found that their period of oscillation is directly related to their intrinsic luminosity.

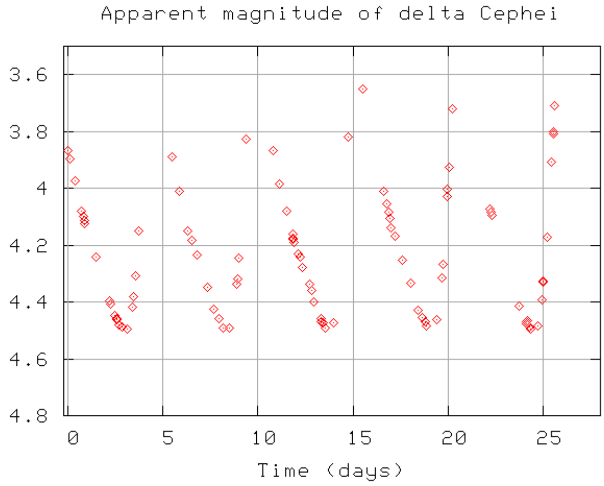

Figure 5: The luminosity of Cepheid stars varies periodically over time.

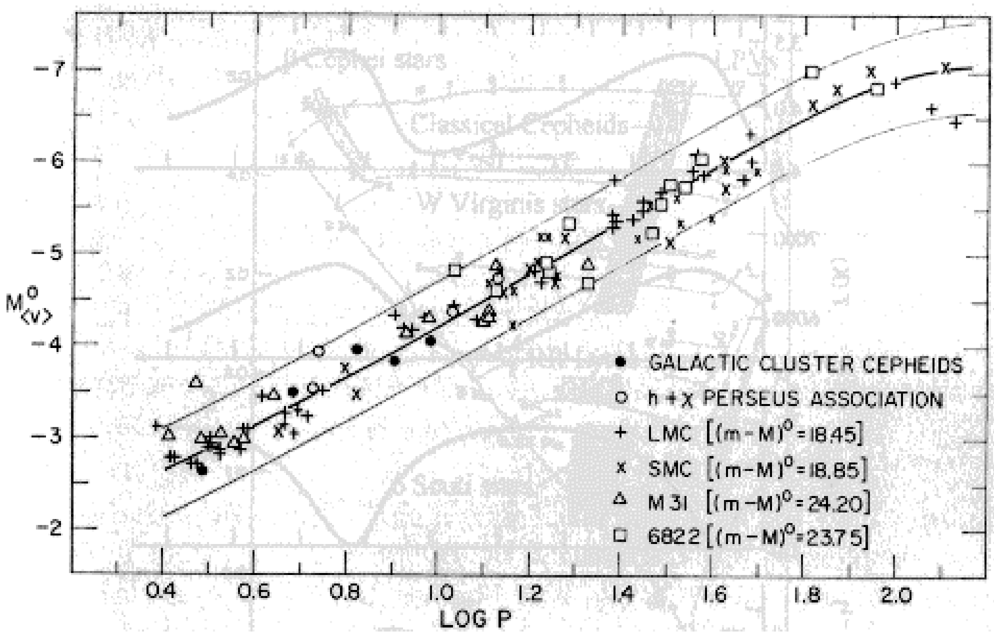

Figure 6: There is a tight relationship between the period (x-axis) and the luminosity (or magnitude in the y-axis) for Cepheid and RRLyrae stars.

These stars pulsate because their surface oscillates up and down like a spring. The gas of the star heats up and then cools down, and the interplay of pressure and gravity keeps it pulsating. How do we know the intrinsic luminosity of these stars? We pick out globular clusters (very bright aglomerations in the Galaxy with about 10^6 stars) and we use spectroscopic parallax to measure their distances. Then we look for the varying stars, measure their brightness and period and build up a plot. There is another class of star called RR Lyrae which also oscillate.

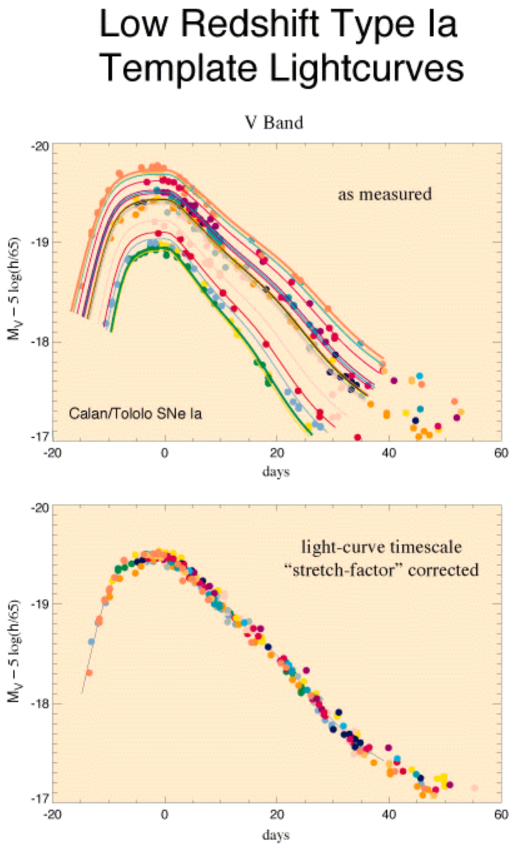

Figure 7: Top panel: the light curves of Supernovae Ia. Bottom panel: the light curves have been recalibrated (or "stretched") so that they all have the same decay rate. Note that, following this procedure, all curves have the same luminosity at the peak.

They have much shorter periods, and are less luminous ($\sim 100\,L_\odot$) but have a much tighter relationship between period and luminosity. We can use these stars to go out to 30Mpc (i.e. 30 million parsecs). If we want to go further, we need something which is even brighter.

The method of choice for measuring very large distances is to look for distant *supernovae*. As you know, supernovae are the end point of stellar evolution, massive explosions that pump out an incredible amount of energy. Indeed supernovae can be as luminous as the galaxies which host them with luminosities of around $10^9\,L_\odot$. So we can see distant supernovae, measure their brightness and if we know their luminosities, use the inverse square law to measure the distance. A certain type of supernova (supernovae I_a) seem to have very similar behaviours. They don't all have the same luminosities but the rate at which they fade after explosion is intimately tied to the luminosity at the moment of the explosion. So by following the ramp up to the explosion and the subsequent decay it is possible to recalibrate a supernova explosion so that we know its luminosity.

Supernovae Ia arise when a white dwarf which is just marginally heavier than the Chandrasekhar mass gobbles up enough material to become unstable and collapse. The electron degeneracy pressure is unable to hold it up and it collapses in a fiery explosion. Supernovae can be used to measure distances out to a distance of about 1000Mpc. They are extremely rare, one per galaxy per hundred years, so we have to be lucky to find them. However there are 10^9 galaxies to look at so the current practice is to stare at large concentrations of galaxies and wait for an event to erupt. Of order 500 SN have been measured in the past decades.

Finally I want to mention another distance indicator which can be used to measure the distances out to about 100 Mpc. When we look at distant galaxies there is a very useful spectral line to measure. It has a wavelength of 21cm and corresponds to the energy associated with the coupling of the spin of the nucleus (a proton) with the spin of an electron in a Hydrogen atom. If they are aligned, the energy will be higher than if they are anti-aligned. Once again, this line will have a certain width due to the Doppler effect as a result of internal motions in the galaxy. In particular the rotation of the hydrogen will induce a Doppler effect. The faster the rotation, the larger the Doppler effect and the wider the spectral line.

We know, from Newtonian gravity that the rotation is intimately tied to the mass of the galaxy, so the wider the line, the faster the speed of rotation and hence the more massive the galaxy. But the more massive the galaxy, the more stars it should contain and therefore the more luminous it should be. So by measuring the 21cm line it is possible to measure the luminosity of distant galaxies. This is known as the *Tully Fisher* relation.

We can now use these techniques to pin down various properties of our Universe.

7 The Thermal history of the Universe: Equilbrium

We shall now look at how the contents of the Universe are affected by expansion. The first property which we must consider is that as the Universe expands, its contents cool down. How can we see that? Let us focus on the radiation contained in the Universe. In the previous sections we found that the energy density in radiation decreases as

$$\rho \propto \frac{1}{a^4}.$$

What else can we say about radiation? Let us make a simplifying assumption, that it is in thermal equilibrium and therefore behaves like a blackbody. For this to be true, the radiation must interact very efficiently with itself to redistribute any fluctuations in energy and occupy the maximum entropy state. You have studied the properties of radiation (or relativistic particles) in thermal equilibrium in statistical mechanics in the 2^{nd} year and found that it can be described in terms of an *occupation number per mode* given by

$$F(\nu) = \frac{2}{\exp \frac{h\nu}{k_B T} - 1}$$

where ν is the frequency of the photon. This corresponds to an *energy density per mode*

$$\epsilon(\nu) d\nu = \frac{8\pi\nu^3 d\nu}{c^3} \frac{h}{\exp \frac{h\nu}{k_B T} - 1}$$

If we integrate over all frequencies we have that the energy density in photons is:

$$\rho_\gamma = \frac{\pi^2}{15}(k_B T) \left(\frac{k_B T}{\hbar c} \right)^3 . \qquad (6)$$

We have therefore that $\rho_\gamma \propto T^4$. Hence if radiation is in thermal equilibrium we have that

$$T \propto \frac{1}{a}$$

Is this the temperature of the Universe? Two ingredients are necessary. First of all, everything else has to feel that temperature which means they have to interact (even if only indirectly) with the photons. For example the scattering off photons of electrons and positrons is through the emission and absorption of photons. And once again, at sufficiently high temperatures, everything interacts quite strongly.

Another essential ingredient is that the radiation must dominate over the remaining forms of matter in the Universe. We have to be careful with this because we know that different types of energy will evolve in different ways as the Universe expands. For example we have that the energy density of dust (or non-relativistic matter) evolves as $\rho_{NR} \propto a^{-3}$ as compared to $\rho_\gamma \propto a^{-4}$ so even if ρ_γ was dominant at early times it may be negligible today. However we also know that the *number density* of photons, $n_\gamma \propto a^{-3}$ as does the number density of non-relativistic particles, $n_{NR} \propto a^{-3}$. If we add up all the non-relativisitic particle in the form of neutrons and protons (which we call baryons), we find that number density of baryons, n_B is very small compared to the number density of photons. In fact we can define the *baryon to entropy ratio*, η_B:

$$\eta_B = \frac{n_B}{n_\gamma} \simeq 10^{-10}$$

As we can see there are many more photons in the Universe than particles like protons and neutrons. So it is safe to say that the temperature of the photons sets the temperature of the Universe.

We can think of the Universe as a gigantic heat bath which is cooling with time. The temperature decreases as the inverse of the scale factor. To study the evolution of matter in the Universe we must now use statistical mechanics to follow the evolution of the various components as the temperature decreases. Let us start off with an ideal gas of bosons or fermions. Its occupation number per mode (now labeled in terms of momentum $\mathbf{p}$) is

$$F(\mathbf{p}) = \frac{g}{\exp\left(\frac{E-\mu}{k_B T}\right) \pm 1}$$

where g is the degeneracy factor, $E = \sqrt{p^2c^2 + M^2c^4}$ is the energy, μ is the chemical potential and + (-) corresponds to the Fermi-Dirac (Bose-Einstein) distribution. We can use this expression to calculate some macroscopic quantities such as the number density

$$n = \frac{g}{h^3}\int \frac{d^3p}{\exp\left(\frac{E-\mu}{k_B T}\right) \pm 1}$$

the energy density

$$\rho c^2 = \frac{g}{h^3}\int \frac{E(\mathbf{p})d^3p}{\exp\left(\frac{E-\mu}{k_B T}\right) \pm 1}$$

and the pressure

$$P = \frac{g}{h^3}\int \frac{p^2c^2}{3E}\frac{d^3p}{\exp\left(\frac{E-\mu}{T}\right)\pm 1}$$

It is instructive to consider two limits. First of all let us take the case where the temperature of the Universe corresponds to energies which are much larger than the rest mass of the individual particles, i.e. $k_BT \gg Mc^2$ and let us take $\mu \simeq 0$. We then have that the number density obeys

$$\begin{aligned} n &= \frac{\zeta(3)}{\pi^2}g\left(\frac{k_BT}{\hbar c}\right)^3 \quad \text{(B.E.)} \\ n &= \frac{3\zeta(3)}{4\pi^2}g\left(\frac{k_BT}{\hbar c}\right)^3 \quad \text{(F.D.)} \end{aligned}$$

where $\zeta(3) \simeq 1.2$ comes from doing the integral. The energy density is given by

$$\begin{aligned} \rho c^2 &= g\frac{\pi^2}{30}(k_BT)\left(\frac{k_BT}{\hbar c}\right)^3 \quad \text{(B.E.)} \\ \rho c^2 &= \frac{7}{8}g\frac{\pi^2}{30}(k_BT)\left(\frac{k_BT}{\hbar c}\right)^3 \quad \text{(F.D.)} \end{aligned}$$

and pressure satisfies $P = \rho c^2/3$. As you can see these are the properties of a radiation. In other words, even massive particles will behave like radiation at sufficiently high temperatures. At low temperatures we have $k_BT \ll Mc^2$ and for both fermions and bosons the macroscopic quantities are given by:

$$\begin{aligned} n &= g\left(\frac{2\pi}{h^2}\right)^{\frac{3}{2}}(Mk_BT)^{3/2}\exp\left(-\frac{Mc^2}{k_BT}\right) \\ \rho c^2 &= Mc^2 n \\ P &= nk_BT \ll Mc^2 n = \rho. \end{aligned}$$

This last expression tells us that the pressure is negligible as it should be for non-relativistic matter.

This calculation has already given us an insight into how matter evolves during expansion. At sufficiently early times it all looks like radiation. As it cools down and the temperature falls below mass thresholds, the number of particles behaving relativistically decreases until when we get to today, there are effectively only three type of particles which behave relativistically: the three types of neutrinos. We denote the *effective number of relativistic degrees of freedom* by g_* and the energy density in relativistic degrees of freedom is given by

$$\rho = g_*\frac{\pi^2}{30}(k_BT)\left(\frac{k_BT}{\hbar c}\right)^3$$

8 The Thermal history of the Universe: The Cosmic Microwave Background

Until now we have considered things evolving passively, subjected to the expansion of the Universe. But we know that the interactions between different components of matter can be far more complex. Let us focus on the realm of chemistry, in particular on the interaction between one electron and one proton. From atomic physics and quantum mechanics you already know that an electron and a proton may bind together to form a Hydrogen atom. To tear the electron away we need an energy of about $13.6eV$. But imagine now that the universe is sufficiently hot that there are particles zipping around that can knock the electron out of the atom. We can imagine that at high temperatures it will be very difficult to keep electrons and protons bound together. If the temperature of the Universe is such that $T \simeq 13.6eV$ then we can imagine that there will be a transition between ionized and neutral hydrogen.

We can work this out in more detail (although not completely accurately) if we assume that this transition occurs in thermal equilibrium throughout. Let us go through the steps that lead to the *Saha equation*. Assume we have an equilibrium distribution of protons, electrons and hydrogen atoms. Let n_p, n_e and n_H be their number densities. In thermal equilibrium (with $T \ll M$) we have that the number densities are given by

$$n_i = g_i \left(\frac{2\pi}{h^2}\right)^{\frac{3}{2}} (M_i k_B T)^{\frac{3}{2}} \exp \frac{\mu_i - M_i c^2}{k_B T}$$

where $i = p, n, H$. In chemical equlibrium we have that

$$\mu_p + \mu_e = \mu_H$$

so that

$$n_H = g_H \left(\frac{2\pi}{h^2}\right)^{\frac{3}{2}} (M_H k_B T)^{\frac{3}{2}} \exp \frac{-M_H c^2}{k_B T} \exp \frac{(\mu_p + \mu_e)}{k_B T}$$

We can use the expressions for n_p and n_e to eliminate the chemical potentials and obtain:

$$n_H = n_e n_p \frac{g_H}{g_p g_e} \left(\frac{2\pi}{h^2}\right)^{-\frac{3}{2}} (M_H k_B T)^{\frac{3}{2}} (M_p k_B T)^{-\frac{3}{2}} (M_e k_B T)^{-\frac{3}{2}} \exp \frac{-M_H c^2 + M_p c^2 + M_e c^2}{k_B T}$$

There are a series of simplifications we can now consider: i) $M_p \simeq M_H$, ii) the binding energy is $B \equiv -M_H c^2 + M_p c^2 + M_e c^2 = 13.6eV$, iii) $n_B = n_p + n_H$ iv) $n_e = n_p$ and finally $g_p = g_e = 2$ and $g_H = 4$. So we end up with

$$n_H = n_p^2 (M_e k_B T)^{-\frac{3}{2}} \left(\frac{2\pi}{h^2}\right)^{-\frac{3}{2}} \exp \frac{B}{k_B T}$$

We can go further and define an ionization fraction $X \equiv \frac{n_p}{n_B}$. Quite clearly we have X is 1 if the Universe is completely ionized and 0 if it is neutral. Using the definition of the baryon to entropy fraction we have

$$1 - X = X^2 \eta_B n_\gamma \left(\frac{2\pi}{h^2}\right)^{-\frac{3}{2}} (M_e k_B T)^{-\frac{3}{2}} \exp \frac{B}{k_B T} \qquad (7)$$

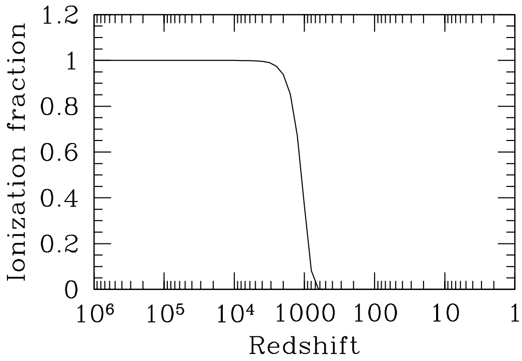

Figure 8: The evolution of the ionization fraction as a function of redshift

Finally we have that we are in thermal equilibrium so we have an expression for n_γ and we get

$$\frac{1-X}{X^2} \simeq 3.8\eta_B \left(\frac{k_B T}{M_e c^2}\right)^{\frac{3}{2}} \exp\frac{B}{k_B T} \tag{8}$$

This is the Saha equation. It tells us how the ionization fraction, X evolves as a function of time. At sufficiently early times we will find that $X = 1$, i.e. the Universe is completely ionized. As it crosses a certain threshold, electrons and protons combine to form Hydrogen. This happens when the temperature of the Universe is $T \simeq 3570K$ or $0.308eV$, i.e. when it was approximately $380,000$ years old, at a redshift of $z \simeq 1100$. We would naively expect this to happen at $13.6eV$ but the prefactors in front of the exponential play an important role. One way to think about it is that, at a given temperature there will always be a few photons with energies larger than the average temperature. Thus energetic photons only become unimportant at sufficiently low temperatures.

What does this radiation look like to us? At very early times, before recombination, this radiation will be in thermal equilibrium and satisfy the Planck spectrum:

$$\rho(\nu)d\nu = \frac{8\pi h}{c^3}\frac{\nu^3 d\nu}{\exp(h\nu/k_B T) - 1}$$

After recombination, the electrons and protons combine to form neutral hydrogen and the photons will be left to propagate freely. The only effect will be the redshifting due to the expansion. The net effect is that the shape of the spectrum remains the same, the peak shifting as $T \propto 1/a$. So even though the photons are not in thermal equilibrium anymore, the spectrum will still be that of thermal equilibrium with the temperature $T_0 = 3000^o/1100$ Kelvin, i.e. $T_0 = 2.75^o$ Kelvin.

The history of each individual photon can also be easily described. Let's work backwards. After recombination, a photon does not interact with anything and simply propagates forward at the speed of light. It's path will be a straight line starting off at the time of recombination and ending today. Before recombination, photons are highly interacting with a very dense medium of charged particles, the protons and electrons. This means that they are constantly scattering off particles, performing something akin to a random walk with a very small step length. For all intents and purposes, they are glued to the spot unable to move. So one can think of such a photon's history as starting off stuck at some point in space and, at recombination, being released to propogate forward until now.

We can take this even further. If we look from a specific observing point (such as the Earth or a satellite), we will be receiving photons from all directions that have been travelling in a straight line since the Universe recombined. All these straight lines will have started off at the same time and at the same distance from us-i.e. they will have started off from the surface of a sphere. This surface, known as the *surface of last scattering* is what we see when we look at the relic radiation. It is very much like a photograph of the Universe when it was 380,000 years old.

9 The Thermal history of the Universe: out of equilibirium and Big Bang Nucleosynthesis

We have assumed that the Universe is in thermal equilibrium throughout this process. We have come up with an expression for the ionization fraction which is not completely accurate but qualitatively has the correct behaviour. There is another situation where assuming thermal equilibrium will not only give us the wrong quantitative but also the wrong qualitative result. We shall now look at what happens when the temperature of the Universe is $k_BT \simeq 1MeV$. This corresponds to the energy where nuclear processes play an important role.

The particles we will consider are protons (p) and neutrons (n). These particles will combine to form the nuclei of the elements. For example

$$\begin{aligned} 1p &\rightarrow \text{Hydrogen nucleus} \\ 1p+1n &\rightarrow \text{Deuterium nucleus} \\ 1p+2n &\rightarrow \text{Tritium nucleus} \\ 2p+1n &\rightarrow \text{Helium 3 nucleus} \\ 2p+2n &\rightarrow \text{Helium 4 nucleus} \end{aligned} \tag{9}$$

The binding energy of each nucleus will be the difference between the mass of the nucleus and the sum of masses of the protons and neutrons that form it. For example the binding energy of Deuterium is $B_D = 2.22MeV$. The neutrons and protons can convert into each other through the weak interactions:

$$\begin{aligned} p+\bar{\nu}_e &\rightarrow n+e^+ \\ p+e^- &\rightarrow n+\nu \\ n &\rightarrow p+e^-+\bar{\nu} \end{aligned} \tag{10}$$

Let us start with an equilibrium approach. We can try to work out the abundance of light elements of the Universe using the same rationale we used above. Once again we have

$$n_i = g_i \left(\frac{2\pi}{h^2}\right)^{\frac{3}{2}} (M_i k_B T)^{\frac{3}{2}} \exp \frac{\mu_i - M_i c^2}{k_B T}$$

Let us assume thermal equilibrium ($\mu_n = \mu_p$). We have that

$$\frac{n_n^{eq}}{n_p^{eq}} = \exp\left(-\frac{Q}{k_B T}\right) \tag{11}$$

where Q is the energy due to the mass difference between the neutron and the proton, $Q = 1.293 MeV$. We can see that at high temperatures there are as many protons as neutrons. But as T falls below Q the mass difference becomes important and the neutrons dwindle away. If this were the correct way of calculating the abundance of neutrons we would find that as the Universe cools, all the neutrons would dissapear. No neutrons would be left.

To get an accurate estimate we must go beyond the equilibrium approximation. We can step back a bit and think about what is actually going on when protons and neutrons are interconverting into each other. The reaction can be characterised in terms of a reaction rate, Γ which has units of s^{-1}. The reaction must compete against the expansion of the universe which itself can be described in terms of a "rate": the expansion rate H which has units of s^{-1}. The relative sizes of Γ and H dictate how important the reactions are in keeping the neutrons and protons equilibrated.

One can write down a Boltzmann equation for the comoving neutron number (the number of neutrons in a box in comoving units):

$$\frac{d\ln N_n}{d\ln a} = -\frac{\Gamma}{H}\left[1 - \left(\frac{N_n^{eq}}{N_n}\right)^2\right]$$

Where $N_n^{eq} = a^3 n_n^{eq}$ is the equilibrium expression given above. If $\Gamma \gg H$ we have that $N_n \to N_n^{eq}$, i.e. the neutron number density will be pushed to its equilibrium value. In that regime we will have the ratio of neutrons as given by 11. If, however, $\Gamma \ll H$, the expansion of the universe will win out and inhibit the depletion or creation of neutrons through that reaction. The equation is then approximately given by

$$\frac{d\ln N_n}{d\ln a} \simeq 0$$

i.e. the neutron comoving number is frozen (and the number density will decay as a^{-3}). The transition from one regime to the other will occur when $\Gamma \sim H$ and it will depend on how the reaction rate depends on temperature and masses. It turns out that for this reaction, the temperature at which reactions "freeze out" is $k_B T_f \simeq 0.7 - 0.8 MeV$. The relative number density of neutrons to protons will be frozen in at

$$\frac{n_n^{eq}}{n_p^{eq}} \simeq \frac{1}{6}$$

In fact the neutron decay rate plays a role as well to further decrease the fraction of neutrons so that we in fact get $n_n/n_p \to \frac{1}{7}$. We can use a very simple argument to find the fraction of Helium 4 in the Universe. We start off with 7/8 in protons and 1/8 in neutrons. Let us assume that the neutrons are used to make Helium atoms. We then need to pair up the 1/8 with 1/8 protons, reducing the number of unpaired protons to $6/8 \simeq 75\%$. So we roughly expect to have about 25% of the mass in Helium and 75% in Hydrogen. A more accurate calculation gives a

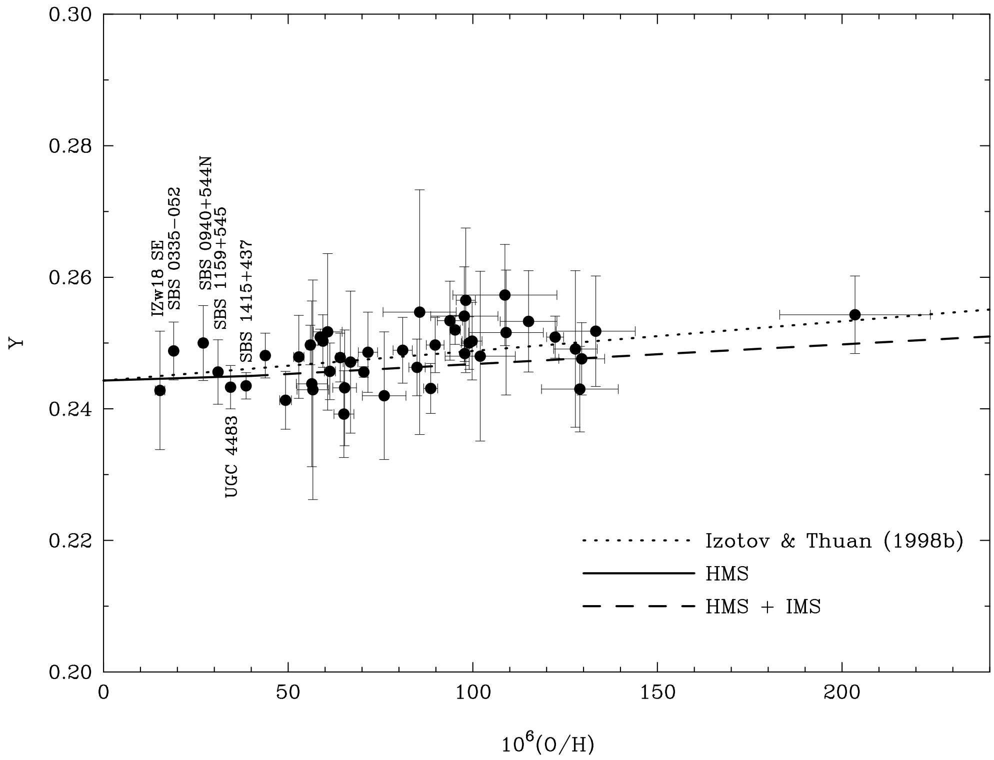

Figure 9: The mass fraction of Helium in stars as a function of the fraction of Oxygen (Izotov & Thuan). The more Oxygen there is, there more stellar burning has occured and therefore more helium has been produced from hydrogen burning. This means that we expect to find a higher amount of helium than what we have found from primordial nucleosynthesis

helium fraction of about 24% which is borne out by observations. One can look at astrophysical systems and measure the amount of Helium. The more Oxygen there is, the more processing there is (and hence the more Helium has been produced in stellar burning).

The abundances of the other light elements is very small but measurable and predictable. For example

$$\frac{n_D}{n_H} \simeq 3 \times 10^{-5}$$

Throughout the history of the Universe, relics have been left over. A relic bath of photons is left over from when the Universe had a $k_B T \simeq eV$. A relic distribution of light elements is left over from when $k_B T \simeq MeV$. It is conceivable that relic particles are left over from transitions which may have occured at higher energies and temperatures.