Colloquium1 – The Peano Axioms

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Abstract

A summary of some notions from Paul Halmos' book *Naive Set Theory*.

1 Introduction

The "Peano Axioms", introduced in 1889 by Peano, and based on earlier work by Dedekind, give an axiomatisation of "natural number". This axiomatization is a list of properties which we expect the "natural numbers" (denoted by $\mathbb{N}$) to have in order to build upon them the rest of mathematics:
natural numbers $\mathbb{N} = \{0,\, 1,\, 2,\, \dots\} \longrightarrow$
integers $\mathbb{Z} = \{0,\, \pm 1,\, \pm 2,\, \dots\} \longrightarrow$
rationals $\mathbb{Q} = \left\{ \frac{p}{q} \,\middle|\, p \in \mathbb{Z} \wedge q \in \mathbb{N} \setminus \{0\} \wedge p \text{ and } q \text{ have no mutual factors} \right\} \longrightarrow$
reals $\mathbb{R}$ (all numbers, including π, e, $\sqrt{2}$ and so on)$\longrightarrow$
analysis $\longrightarrow$
geometry $\longrightarrow$
physics

1.1 The Peano Axioms

1. Zero is a natural number:
   $\boxed{0 \in \mathbb{N}}$

2. If n is a natural number, so is $n+1$:
   $\boxed{n \in \mathbb{N} \Longrightarrow (n+1) \in \mathbb{N}}$

3. The induction principle: If a statement $S(n)$ is true for $n = 0$ and $[S(n+1)$ is true whenever $S(n)$ is true$]$ for all $n \in \mathbb{N}$, then $S(n)$ is true for all natural numbers n. (also called minimal property):
   $\boxed{(S(0) \wedge [(S(n) \Longrightarrow S(n+1))\, \forall n \in \mathbb{N}]) \Longrightarrow S(n)\, \forall n \in \mathbb{N}}$

4. For any natural number n, $n+1 = 0$ is false:
   $\boxed{(n+1 \neq 0)\, \forall n \in \mathbb{N}}$

5. For any two natural numbers n and m, if $n+1 = m+1$ then $n = m$:
$\boxed{\forall n \in \mathbb{N}, \forall m \in \mathbb{N} : ((n+1 = m+1) \Longrightarrow (n = m))}$

These axioms were chosen to list all the facts we expect to be true about the natural numbers. Two questions arise: does there exist a mathematical rigorous object that obeys these conditions, and if so, is it unique?

The answer to these questions is given by set theory.

2 Construction of the Axioms as Theorems from Set Theory

Our aim is to define the natural numbers out of sets, and see how from the basic axioms of sets we can prove the Peano Axioms as mere theorems. Having done that, we will have been able to give a foundation for a huge corpus of mathematics in terms of sets.

Note: It is a general goal of mathematics to ultimately convert all assumptions of higher-level mathematics to theorems that depend on the axioms of set theory (or some other foundation of mathematics).

2.1 Definition of the Natural Numbers within Set Theory

- For every set x, define the succssor set of the set x (Nachfolge), denoted by x^+ as
  $\boxed{x^+ := x \cup \{x\}}$.
  - Example: Assume $x = \{1, 2, 3, 4\}$. Then $x^+ = x \cup \{x\} = \{1, 2, 3, 4, x\} = \{1, 2, 3, 4, \{1, 2, 3, 4\}\}$. Note that then $|x| = 4$ and $|x^+| = |x| + 1 = 5$. We always have that $|x^+| = |x| + 1$.
- Define 0 to be *the* set with zero number of elements. We use the definite article "the" because there is only one such set (by the axiom of extension), namely, the empty set.
  $\boxed{0 := \varnothing}$.
- Define the rest of the natural numbers as succesors of 0:
  - $\boxed{1 := 0^+ = \varnothing \cup \{\varnothing\} = \{\varnothing\} = \{0\}}$
  - $\boxed{2 := 1^+ = \{\varnothing\} \cup \{\{\varnothing\}\} = \{\varnothing, \{\varnothing\}\} = \{0, 1\}}$
  - $\boxed{3 := 2^+ = \{\varnothing, \{\varnothing\}\} \cup \{\{\varnothing, \{\varnothing\}\}\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} = \{0, 1, 2\}}$
  - and so on.
- Define a successor set as a set such that one of its elements is the empty set and for every one of its elements, it also contains the successor set of that element.

That is, for a given set, A, to be "a successor set", the following two conditions must be fulilled:

1. $\varnothing \in A$
2. $x^+ \in A$ for all $x \in A$.

- Recall (or find out) $\boxed{\text{The "axiom of infinity": there exists a successor set.}}$
This is one of the axioms of set theory, the underlying basis of mathematics. It says that in the world of all possible sets, there *is* one set that obeys the two conditions we stipulated just above. It doesn't say what the set is, just that there is one. It is easy to see why this is called the axiom of infinity.

- *Claim:* The non-empty intersection of every family of successor sets is a successor set itself.
Proof:

  - Let $\{A_i\}_{i \in I}$ be a family of successor sets. That is, $\{A_1, A_2, A_3, \dots\}$ if $I = \{1, 2, 3, \dots\}$, where each A_i is "a successor set".
  We want to show that $\cap_{i \in I} A_i \equiv A_1 \cap A_2 \cap A_3 \cap \dots$ is a successor set.
  - Since A_i is a successor set for each $i \in I$, $\varnothing \in A_i$ for each $i \in I$. As a result, $\varnothing \in \cap_{i \in I} A_i$.
  - Take some $x \in \cap_{i \in I} A_i$. That means that $x \in A_i$ for all $i \in I$. Since each A_i is a successor set for all $i \in I$, $x^+ \in A_i$ for all $i \in I$. As a result, $x^+ \in \cap_{i \in I} A_i$.

  $\blacksquare$

- Let A_0 be some arbitrary successor set (we know one exists by the axiom of infinity).

- Define $\mathbb{N}$, the natural numbers, as the intersection of all successor sets which are subsets of A_0. From the above that means that $\mathbb{N}$ is itself a successor set.

- Note that this definition sounds crazy for now, because why should a universal important object such as $\mathbb{N}$, depend on our choice of A_0? We will see that in fact it does not, and the same result would be for any A_0 we choose.

- In symbols we have

$$\boxed{\mathbb{N} := \bigcap_{X \text{ is a successor set that is a subset of } A_0} X}$$

- *Claim:* $\mathbb{N} \subset B_0$ for *any* other successor set B_0.
Proof:

- Let B_0 be a given successor set.
- Since A_0 and B_0 are successor sets, $A_0 \cap B_0$ is also a successor set by the above claim.
- $A_0 \cap B_0 \subset A_0$ is always true for any two sets.
- Thus, $A_0 \cap B_0$ is a successor set which is a subset of A_0. But this subset of A_0 also goes into the definition of $\mathbb{N}$.
- As a result, $\mathbb{N}$ must be a subset of $A_0 \cap B_0$ according to its definition: $\mathbb{N} \subset A_0 \cap B_0$.
- But from $\mathbb{N} \subset A_0 \cap B_0$ it necessarily follows as well that $\mathbb{N} \subset B_0$.

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- Thus we see that $\mathbb{N}$ indeed does *not* depend on A_0, because we find $\mathbb{N}$ to the smallest successor set: it is a successor set with the property that it is a subset of every other successor set!

- *Claim:* $\mathbb{N}$ is unique.
Proof:

  - Assume otherwise, that is, assume ψ is another successor set that is included in any possible successor set, just like $\mathbb{N}$.
  - Since $\mathbb{N}$ is a successor set, $\psi \subset \mathbb{N}$.
  - But $\mathbb{N}$ is *also* "a successor set that is included in any other successor set", so that $\mathbb{N} \subset \psi$.
  - As a result we see that $\mathbb{N}$ and ψ have exactly the same elements.
  - Recall (or find out)
  $\boxed{\text{The axiom of extension: Two sets are equal iff they have the same elements.}}$
  - Thus we have $\psi = \mathbb{N}$, and so, *any* successor set with the property that "it is included in any other successor set" has to be equal to $\mathbb{N}$.

$\blacksquare$

- Define a "natural number" to be an element in $\mathbb{N}$, and "the natural numbers" as $\mathbb{N}$.

- Note: we pay a price for a set-theoretic definition of natural numbers: we obtain "superfluous" structure which say, for instance, that not only $7^+ = 8$, but also that $7 \in 8$: a very peculiar fact.

2.2 The "Axioms" Follow from Set Theoretic Definitons

2.2.1 The First "Axiom"

- *Claim:* Zero is a natural number.
Proof:

- Since $\mathbb{N}$, the set of natural numbers is a successor set, and successor sets contain $\varnothing$, $\varnothing \in \mathbb{N}$.
- However, $0 = \varnothing$, so that $0 \in \mathbb{N}$, as we had hoped.

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2.2.2 The Second "Axiom"

- *Claim:* If n is a natural number, so is $n + 1$.
  Proof:
  - We identify $n + 1$ as n^+.
  - Since n is a natural number, $n \in \mathbb{N}$.
  - But since $\mathbb{N}$ is a successor set, that means that $n^+ \in \mathbb{N}$, that is, $(n + 1) \in \mathbb{N}$.

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2.2.3 The Third Axiom – The Principle of Mathematical Induction

- *Claim:* If a statement $S(n)$ is true for $n = 0$ and $[S(n+1)$ is true whenever $S(n)$ is true$]$ for all $n \in \mathbb{N}$, then $S(n)$ is true for all natural numbers n.
  Proof:
  - Define the set $X := \{\, n \in \mathbb{N} \mid S(n) \text{ is true} \,\}$.
  - *Claim:* X is a successor set.
    Proof:
    * We are given that $S(0)$ is true, and so $0 \in X$.
    * We are given that $S(n+1)$ is true if $S(n)$ is true for all $n \in \mathbb{N}$. This means that $(n + 1) \in X$ if $n \in X$ for all $n \in \mathbb{N}$.
    * Thus we have proven that X is a successor set.
  - But $\mathbb{N}$ is a successor set that is a subset of any other successor set (minimality property), which means that $\mathbb{N} \subset X$. But by the definition of X, $X \subset \mathbb{N}$.
  - Thus by the extension axiom, $X = \mathbb{N}$ and so the statement $S(n)$ is true for all $n \in \mathbb{N}$.

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2.2.4 The Fourth Axiom

- *Claim:* For any natural number n, $n + 1 = 0$ is false.
  Proof:
  - Note that $0 = \varnothing$, that is, 0 is a set containing no elements.

- $n+1 \equiv n^+ \equiv n \cup \{n\}$, that is, $n \in n^+$. Thus, necessarily, n^+ *has* an element, and is not empty.
- By the axiom of extension $n^+ \neq 0$.
- Since n was arbitrary this is true for all $n \in \mathbb{N}$.

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2.2.5 The Fifth Axiom

- *Claim:* For any two natural numbers n and m, if $n + 1 = m + 1$ then $n = m$.
  Proof:
  - *Claim 1:* No natural number is a subset of any of its elements. In symbols: $\forall n \in \mathbb{N}\ (\forall m \in n\ (n \not\subseteq m))$.
    Proof:
    * Define $S := \{\, n \in \mathbb{N} \mid \forall m \in n\, (n \not\subseteq m) \,\}$.
    * *Claim:* $S = \mathbb{N}$.
      Proof:
      · $0 = \varnothing$, so $0 \in S$ because the condition holds vacuously (there *are* no $m \in 0$ to violate the condition).
      · Let some $n \in S$ be given. We want to show now that $n^+ \in S$, that is, $\forall m \in n^+$, $n^+ \not\subseteq m$.
      · $n^+ \equiv \{1, 2, \ldots, n\} = n \cup \{n\}$.
      · *Case 1–$m = n$*: $n \subset n$ trivially, so that we *cannot* have $n \in n$ (otherwise n would not have been in S). But $n \in n^+$, so that we found some object (n) which is in n^+ and not in n, and thus $n^+ \not\subseteq n$.
      · *Case 2–$m \in \{1, 2, \ldots n-1\}$*: Then assume otherwise, that $n^+ \subset m$. Then also $n \subset m$ because $n^+ \equiv n \cup \{n\}$. But $n \in S$, so that means that $m \notin n$. This is a contradiction as we know that $m \in \{1, 2, \ldots, n-1\} \equiv n$. Thus it must be the case that $n^+ \not\subseteq m$.
      · Thus we have by the third axiom that $S = \mathbb{N}$.
  - *Claim 2:* Every element of a natural number is a subset of it. In symbols: $\forall n \in \mathbb{N}\ (\forall m \in n\ (m \subset n))$.
    Proof:
    * Define $S := \{\, n \in \mathbb{N} \mid \forall m \in n\, (m \subset n) \,\}$.
    * *Claim:* $S = \mathbb{N}$.
      Proof:
      · $0 \in S$ because the condition is vacuously satisfied for $0 = \varnothing$ which has no elements.
      · Let some $n \in S$ be given. We want to show now that $n^+ \in S$, that is, $\forall m \in n^+$, $m \subset n^+$.

        - $n^+ \equiv \{1, 2, \ldots, n\} = n \cup \{n\}$.
        - *Case 1–*$m \in \{1, 2, \ldots, n-1\}$*:* Then $m \in n$ and because $n \in S$, $m \subset n$. But $n \subset n^+(n^+ \equiv n \cup \{n\})$ so that $m \subset n^+$.
        - *Case 2–*$m = n$*:* By definition, $n \subset n^+$, and so $m \subset n^+$.
        - Thus we found that $n^+ \in S$.
        - Using the third axiom we have that $S = \mathbb{N}$.
    - Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$ be given and further assume that $n^+ = m^+$.
    - Because $n \in n^+$, $n \in m^+$. But $m^+ = m \cup \{m\}$, so that either $n \in m$ or $n = m$.
    - By a symmetric argument we have that either $m \in n$ or $m = n$.
    - Assume $n \neq m$. Then the two arguments above imply that both $n \in m$ and $m \in n$ are true.
    - By *Claim 2,* every element of a natural number is a subset of it. $m \in n$ where n is a natural number. So we have $m \subset n$. But $m \subset n$ and $n \in m$ implies that $n \in n$. But this is of course a contradiction using *Claim 1* because we *always* have that $n \subset n$ (for every set) and so $n \in n$ and $n \subset n$ imply a contradiction with *Claim 1.*
    - As a result we *must* have $n = m$.

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3 A Few More Theorems (Homework)

- *Claim:* $(n \neq n^+) \, \forall n \in \mathbb{N}$.
- *Claim:* $\forall n \in \mathbb{N}(n \neq 0 \implies n = m^+$ for some $m \in \mathbb{N})$.
- *Claim:* Every element of a natural number is a subset of it: $\forall m \in \mathbb{N}$ $(m \subset \mathbb{N})$.
- *Claim:* $\forall n \in \mathbb{N}$ ($\forall E \subset n$ such that $E \neq \varnothing$ ($\exists k \in E$ such that ($k \in m$ whenever $m \in E \setminus \{k\}$))).

4 Next Steps

Arithmetic and Order: chapters 13 and 14.